Elliptic Curve Cryptography

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Terminology

plaintext - original message

ciphertext - coded form of the message

enciphering/encryption - converting the plaintext to ciphertext (enciphering key - specific process used to do this)

deciphering/decryption - restoring the plaintext from the ciphertext (deciphering key - specific process used to do this)

cipher - method used to encipher/decipher

cryptanalysis - deciphering a ciphertext message without knowing the cipher

Modular Arithmetic

$\boldsymbol{x} \pmod{\boldsymbol{n}} = \boldsymbol{r}$ where r is the remainder when integer x is divided by n (n is a positive integer and $0 \leq r \leq n-1$).

- $51 \pmod{9} = 6 \qquad (51 = 5 \cdot 9 + 6)$
- $213 \pmod{10} = 3 \qquad (213 = 21 \cdot 10 + 3)$
- $62 + 81 \pmod{11} = 0 \qquad (62 + 81 = 143 = 13 \cdot 11 + 0)$
- $23^5 \pmod{26} = 17 \qquad (23^5 = 6436343 = 247551 \cdot 26 + 17)$
- $13 \pmod{26} = 13 \qquad (13 = 0 \cdot 26 + 13)$

Numerical Equivalents

letter:	A	B	C	D	E	F	G	H	I	J	K	L	M
NE:	00	01	02	03	04	05	06	07	08	09	10	11	12
letter:	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
NE:	13	14	15	16	17	18	19	20	21	22	23	24	25

LEAVING ON THE 1330 TRAIN TO PARIS.

1104002108130614131907041330191700081319141500170818

11040 02108 13061 41319 07041 33019 17000 81319 14150 01708 18999

More Modular Arithmetic

Given a positive integer n and an integer x where $1 \leq x \leq n-1$, the integer x has an **inverse** $\pmod{n}$ if there exists an integer y, with $1 \leq y \leq n-1$, such that $xy = 1 \pmod{n}$. In this case, $y = x^{-1} \pmod{n}$.

$3^{-1} \pmod{26} = 9, \qquad 3 \cdot 9 \pmod{26} = 27 \pmod{26} = 1$

$$\begin{aligned} 4 \cdot 6 \pmod{26} &= 24 \pmod{26} = 24 \\ 4 \cdot 7 \pmod{26} &= 28 \pmod{26} = 02 \\ 4 \cdot 12 \pmod{26} &= 48 \pmod{26} = 22 \\ 4 \cdot 13 \pmod{26} &= 52 \pmod{26} = 00 \end{aligned}$$

4 has no inverse $\pmod{26}$.

Problem

- Alice and Bob need a secure key exchange or need to share messages over a line watched by Eve. Must meet or communicate the details of the cipher (or message) over an insecure channel.

Solution

- James Ellis (1969) - worked for the British Government Communications Headquarters - not declassified (1997) until after his death. His idea was to add "noise" to create the ciphertext and then subtract it to get the plaintext.
- Whitefield Diffie & Martin Hellman (1976) - published "New Directions in Cryptography" after Diffie met with Ellis.

Symmetric Ciphers: Both Alice and Bob share or have equal knowledge of the secret key used to encrypt and decrypt messages.

Asymmetric or Public Key Ciphers: The key used to encrypt is distinct from the key used to decrpyt and computing the deciphering method from the enciphering method is not feasible.

Public Key: Method used to encipher messages. This is created by Bob and anyone (even Eve) can know it.

Private Key: Method used to decipher the messages. This is also created by Bob and remains with him.

The Discrete Logarithm Problem

The field $\mathbb{F}_p^*$ is the collection of elements $\{1, 2, \ldots, p-2, p-1\}$ where if $x, y \in \mathbb{F}_p^*$, then $xy = xy \pmod{p}$.

Let p be a prime number. Then there exists an element $g \in \mathbb{F}_p^*$ whose powers give every element of $\mathbb{F}_p^*$, i.e.

$$\mathbb{F}_p^* = \{1, g, g^2, \ldots, g^{p-2}, g^{p-1}\}$$

Elements with this property are called **primitive roots** of $\mathbb{F}_p^*$ or **generators** of $\mathbb{F}_p^*$.

The Discrete Logarithm Problem

$\mathbb{F}_{11}^*$ has 2 as a primitive root.

$$2^0 = 1 \quad 2^1 = 2 \quad 2^2 = 4 \quad 2^3 = 8 \quad 2^4 = 5$$
$$2^5 = 10 \quad 2^6 = 9 \quad 2^7 = 7 \quad 2^8 = 3 \quad 2^9 = 6$$

However, 2 is not a primitive root for $\mathbb{F}_{17}^*$.

$$2^0 = 1 \quad 2^1 = 2 \quad 2^2 = 4 \quad 2^3 = 8 \quad 2^4 = 16$$
$$2^5 = 15 \quad 2^6 = 13 \quad 2^7 = 9 \quad 2^8 = 1$$

The Discrete Logarithm Problem

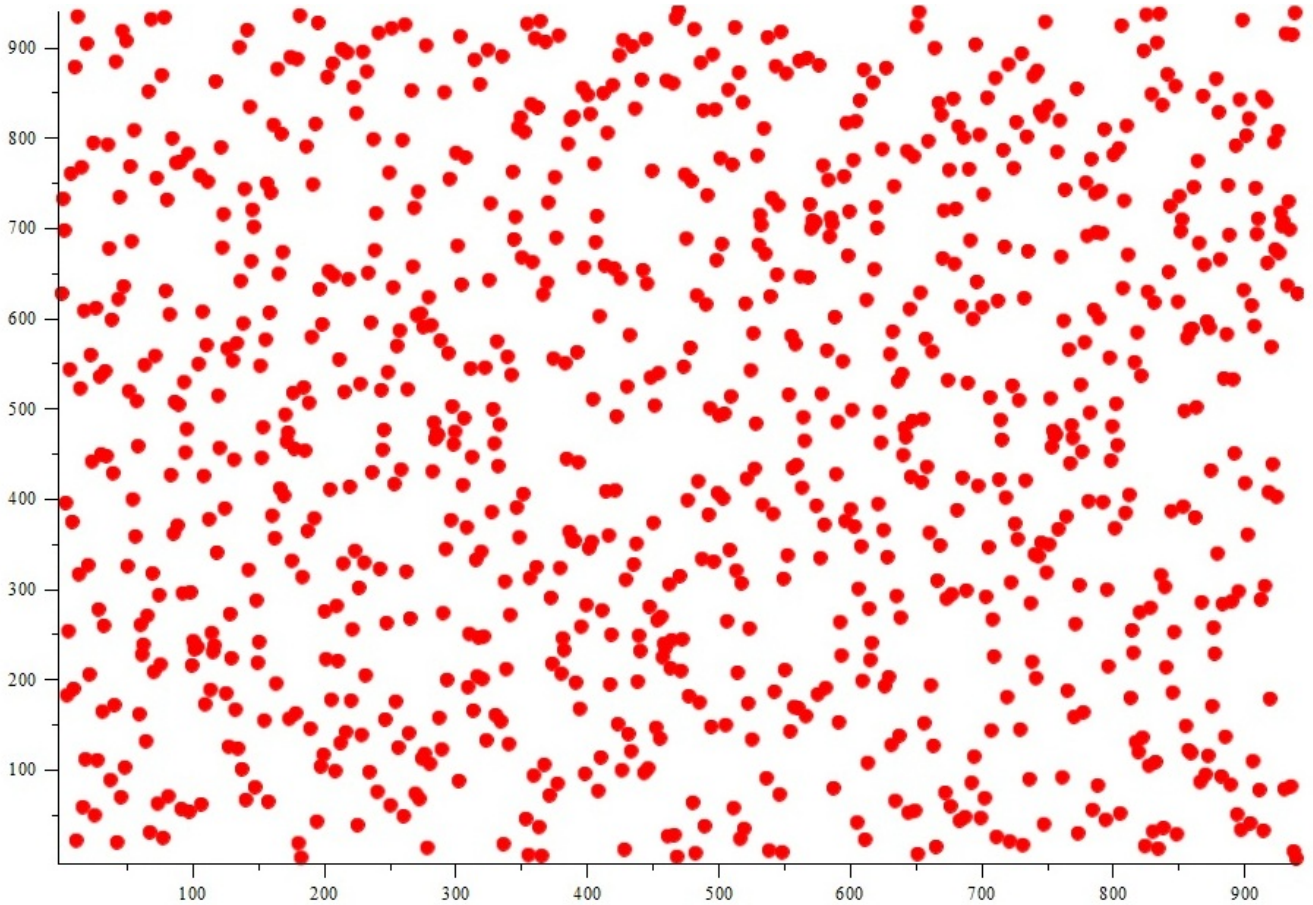
Let g be a primitive root for $\mathbb{F}_p^*$ and let h be an integer in $\mathbb{F}_p^*$. The **Discrete Logarithm Problem (DLP)** is the problem of finding an exponent x such that

$$g^x = h \pmod{p}.$$

Given $\mathbb{F}_{941}^*$ with primitive root 627, only real way to solve the DLP $627^x = 551 \pmod{941}$ is to compute $627^1, 627^2, 627^3, \ldots$ until you get $627^{817} = 551 \pmod{941}$.

This is a **hard** problem.

The Discrete Logarithm Problem



Powers of $627^i \pmod{941}$ for $i = 1, 2, 3, \ldots$

Diffie-Hellman Key Exchange

Alice and Bob first agree on a large prime number p and an integer g, where $1 \leq g \leq p-1$, and make them public.

Alice picks a secret integer a - Bob picks a secret integer b.

$\underbrace{A = g^a \pmod{p}}_{\text{Alice computes this}}$ and $\underbrace{B = g^b \pmod{p}}_{\text{Bob computes this}}$

Exchange Values

$\underbrace{A' = B^a \pmod{p}}_{\text{Alice computes this}}$ and $\underbrace{B' = A^b \pmod{p}}_{\text{Bob computes this}}$

$$A' = B^a = (g^b)^a = g^{ab} = (g^a)^b = A^b = B' \pmod{p}$$

Diffie-Hellman Key Exchange

Prime number $p = 7001$ Base $g = 101$

Alice picks $a = 300$ Bob picks $b = 2512$

$$A = g^a \pmod{p} = 101^{300} \pmod{7001} = 1910$$

$$B = g^b \pmod{p} = 101^{2512} \pmod{7001} = 5533$$

Exchange Values

$$A' = B^a \pmod{p} = 5533^{300} \pmod{7001} = 5161$$

$$B' = A^b \pmod{p} = 1910^{2512} \pmod{7001} = 5161$$

Diffie-Hellman Key Exchange

Eve knows $g, p, A = g^a$, and $B = g^b$. To find the private key, she must solve one of two DLPs.

Find a by solving $g^a = A \pmod{p}$.

Find b by solving $g^b = B \pmod{p}$.

RSA

- Create by Ron Rivest, Adi Shamir, and Len Adleman in 1978 at MIT.
- First public-key cipher.
- Still (as far as I know) widely used today.

RSA

1. Bob generates two distinct large prime numbers p and q. He then computes $m = pq$ and $n = (p-1)(q-1)$.

   $p = 103,\ q = 191$

   $m = pq = 103 \cdot 191 = 19673, n = (p-1)(q-1) = 102 \cdot 190 = 19380$

- p and q should each be of binary length 1024 (309 digits) or larger.
- p and q should approximately be of the same size.
- Here all of the calculations take place in $\mathbb{F}_{pq}^* = \{x | 1 \leq x \leq pq - 1 \text{ and } x \text{ and } pq \text{ are relatively prime}\}$.

RSA

2. Bob selects a number e that is relatively prime to n.

$$\text{Pick } e = 23{\cdot}29 = 667 \qquad n = 19380 = 2^2{\cdot}3{\cdot}5{\cdot}17{\cdot}19$$

3. Bob finds d such that $ed = 1 \pmod{n}$ (use Euclidean Algorithm).

$$d = 523$$

4. Bob makes $e = 667$ and $m = 19763$ public (public key). Security based on the ability to factor m.

RSA

5. Alice arranges message as a series of numbers x such that $0 \leq x \leq m-1$ or $0 \leq x \leq 19672$.

L	E	A	V	E		A	T		N	O	O	N	Z
11	04	00	21	04		00	19		13	14	14	13	25

11040 02104 00191 31414 13259

6. For each number x, Alice computes $y = x^e \pmod{m}$ or $y = x^{667} \pmod{19673}$.

05073 05739 15089 03707 03344

RSA

7. Alice sends the ciphertext to Bob and he deciphers it using his private key d and n. Bob computes $x = y^d \pmod{m}$ or $x = y^{523} \pmod{19673}$.

Comments:

- $x \to x^e \pmod{m} \to (x^e)^d \pmod{m} = x^{ed} \pmod{m} = x$
- As factoring methods improve, need to find larger primes (safe here as there are an infinite number of primes.
- Or, we could make the algebra harder.

A Little Group Theory

Generalize $\mathbb{F}_p^*$ with operation $x \cdot y = xy \pmod{p}$ to a non-abelian (non-commutative) group. A group is a collection of objects with a binary operation $\star$ such that

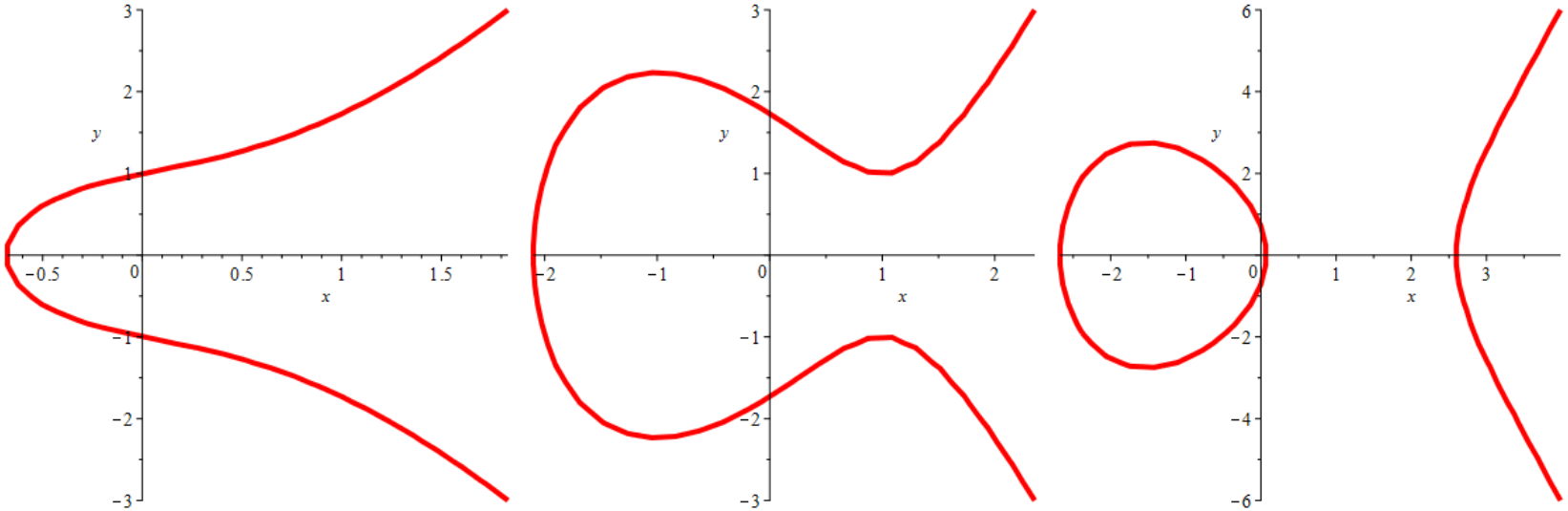
- $(x \star y) \star z = x \star (y \star z)$,
- there exists identity e where $x \star e = e \star x = x$, and
- for each x the element x^{-1} exists such that $x \star x^{-1} = x^{-1} \star x = e$.

Elliptic Curve Cryptography

An **elliptic curve E** is the set of solutions to the equation of the form

$$Y^2 = X^3 + AX + B.$$

together with an extra point $\mathcal{O}$, where the constants A and B satisfy $A^3 + 27B^2 \neq 0$.



You can define an operation (way to add points) on the points of an elliptic curve.

Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be points on the elliptic curve $y^2 = x^3 + Ax + B$.

a) If $P_1 = \mathcal{O}$, then $P_1 + P_2 = P_2$.

b) If $P_2 = \mathcal{O}$, then $P_1 + P_2 = P_1$.

c) If $x_1 = x_2$ and $y_1 = -y_2$, then $P_1 + P_2 = \mathcal{O}$.

d) Otherwise, define λ by

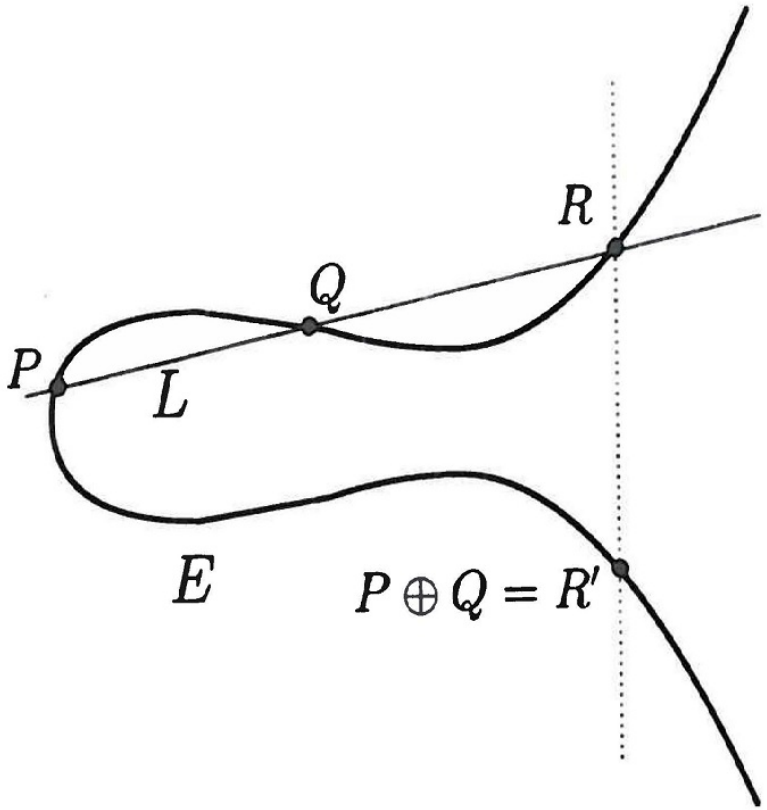
$$\lambda = \begin{cases} \frac{y_2 - y_1}{x_2 - x_1} & \text{if } P_1 \neq P_2 \\ \frac{3x_1^2 + A}{2y_1} & \text{if } P_1 = P_2 \end{cases}$$

and let

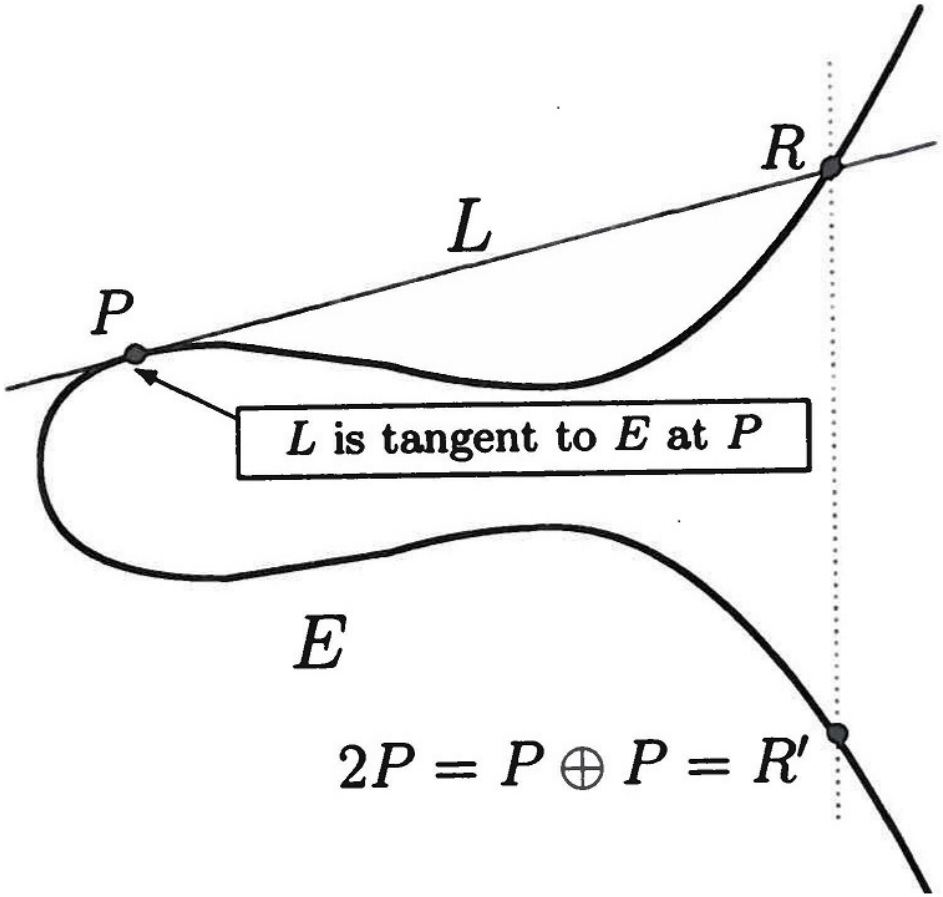
$$x_3 = \lambda^2 - x_1 - x_2 \qquad \text{and} \qquad y_3 = \lambda(x_1 - x_3) - y_1.$$

Then $P_1 + P_2 = (x_3, y_3)$.

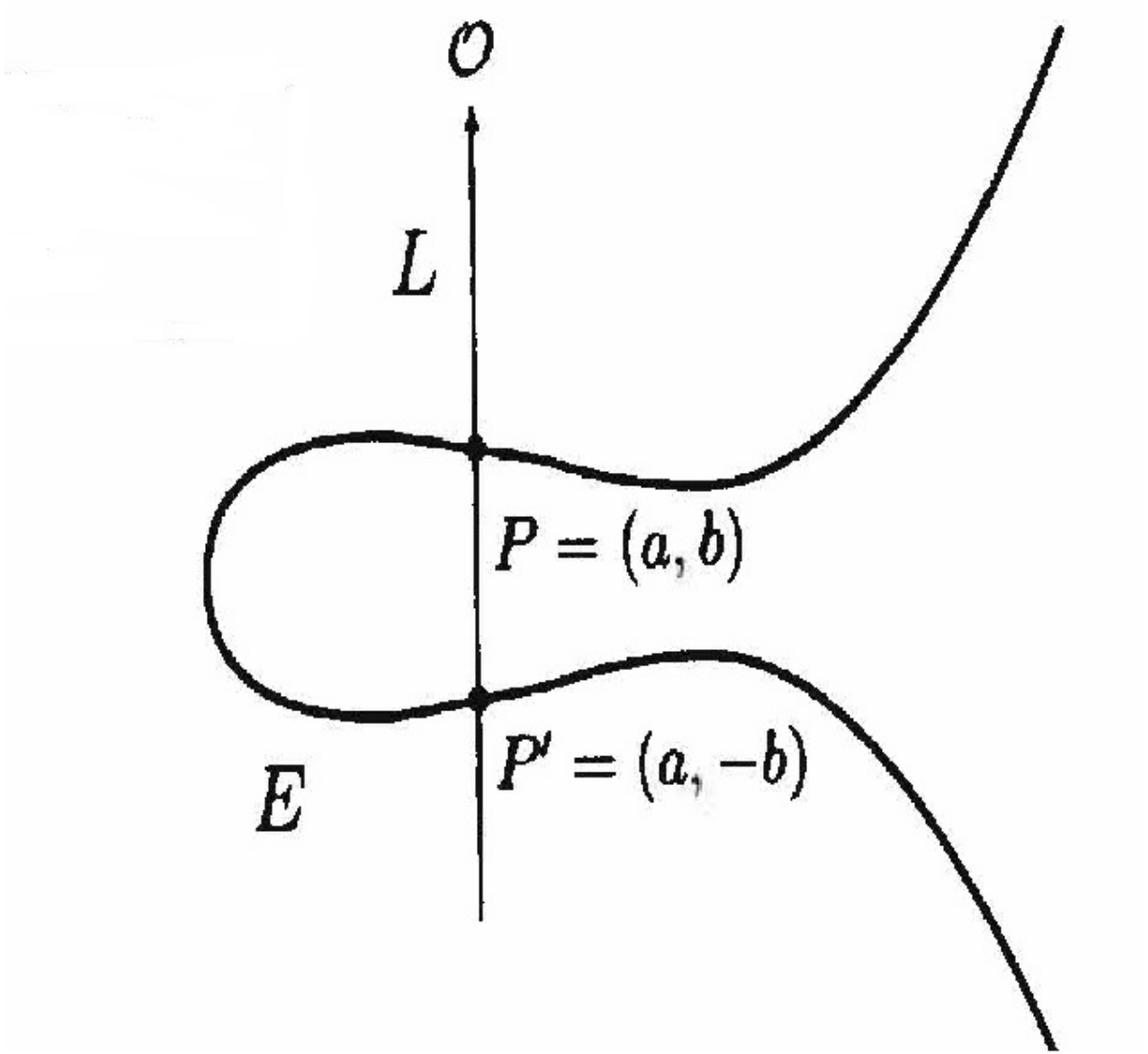
Elliptic Curve Cryptography



Elliptic Curve Cryptography



Elliptic Curve Cryptography



Elliptic Curve over Finite Fields

$$\mathbb{F}_p = \{0, 1, \ldots, p-1\}$$

$$E : Y^2 = X^3 + AX + B \text{ with } A, B \in \mathbb{F}_p$$

$$E(\mathbb{F}_p) = \{(x, y) \mid x, y \in \mathbb{F}_p \text{ satisfying } y^2 = x^3 + Ax + B\} \cup \{\mathcal{O}\}$$

Elliptic Curve over Finite Fields

$\mathbb{F}_{13} = \{0, 1, \ldots, 12\}$

$$E : Y^2 = X^3 + 3X + 8$$

$$E(\mathbb{F}_{13}) = \{\mathcal{O}, (1,5), (1,8), (2,3), (2,10), (9,6), (9.7), (12,2), (12,11)\}$$

$$(1,5) \oplus (9,6) = (2,3) \qquad (12,2) \oplus (2,10) = (9,6)$$

Elliptic Curve Discrete Log Problem (ECDLP)

Let P be a point on $E(\mathbb{F}_p)$. Let n be a positive integer and compute $Q = nP = \underbrace{P \oplus P \oplus \cdots \oplus P}_{n}$. Publish P and Q.

The ECDLP is to find n such that

$$Q = nP.$$

The ECDLP is "harder" that the DLP.

Fastest DLP Solution Method - order of $\log p$.

Fastest ECDLP Solution Method - order of $\sqrt{p}$.

Elliptic Curve Diffie-Hellman Key Exchange

Alice and Bob first agree on a large prime number p, and elliptic curve E, and a point $P \in E(\mathbb{F}_p)$.

Alice picks a secret integer a - Bob picks a secret integer b.

$$\underbrace{Q_1 = aP = (P + \cdots + P)}_{\text{Alice computes this}} \quad \text{and} \quad \underbrace{Q_2 = bP = (P + \cdots + P)}_{\text{Bob computes this}}$$

Exchange Values

$$\underbrace{R' = aQ_2 = (Q_2 + \cdots + Q_2)}_{\text{Alice computes this}} \quad \text{and} \quad \underbrace{S' = bQ_1 = (Q_1 + \cdots Q_1)}_{\text{Bob computes this}}$$

$$R' = aQ_2 = a(bP) = abP = b(aP) = b(Q_1) = S'$$

Elliptic ElGamal Public Key Cryptosystem

1. Alice and Bob agree on p, the elliptic curve E, and point $P \in E(\mathbb{F}_p)$.
2. Bob chooses n_B and publishes $Q_B = n_B P$ as public key.
3. Alice's plaintext is a point $M \in E(\mathbb{F}_p)$.
4. Alice chooses an integer k and computes

$$C_1 = kP \quad \text{and} \quad C_2 = M + kQ_B$$

5. Alice send the two points C_1 and C_2 to Bob.
6. Bob computes the following to obtain the plaintext M.

$$C_2 - n_B C_1 = (M + kQ_B) - n_B(kP) = M + k(n_B P) - n_B(kP) = M$$

Elliptic ElGamal Public Key Cryptosystem

1. Alice and Bob agree on $p = 3851$, the elliptic curve $E : y^2 = x^3 + 324x + 1287$, and point $P = (920, 303) \in E(\mathbb{F}_{3851})$.
2. Bob chooses $n_B = 2489$ and publishes $Q_B = n_B P = 2489(920, 303) = (593, 719)$ as public key.
3. Alice's plaintext is a point $M = (3681, 612) \in E(\mathbb{F}_p)$.
4. Alice chooses an integer $k = 3021$ and computes

$$C_1 = kP = 3021(920, 303) = (343, 3454)$$

and

$$\begin{aligned} C_2 &= M + kQ_B = (3681, 612) + 3021(593, 719) \\ &= (3681, 612) + (252, 3610) = (3506, 686) \end{aligned}$$

Elliptic ElGamal Public Key Cryptosystem

5. Alice send the two points $C_1 = (343, 3454)$ and $C_2 = (3506, 686)$ to Bob.
6. Bob computes the following to obtain the plaintext M.

$$\begin{aligned} M = C_2 - n_B C_1 &= (3506, 686) - 2489(343, 3454) \\ &= (3506, 686) - (252, 3610) \\ &= (3681, 612) \end{aligned}$$