Secure Data Transmission: The Mathematics Behind Identification Numbers and Check Digit Schemes

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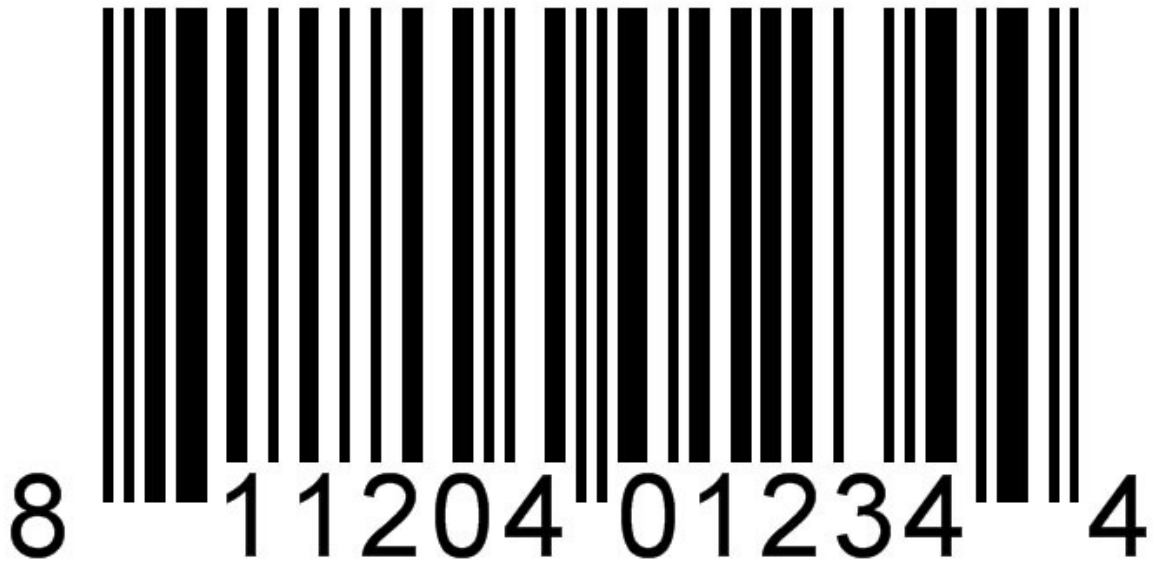
US Postal Money Order

Universal Product Code - UPC



Vehicle Identification Number & State of Washington Driver's License Number

1 G 1 Y Z 2 3 J 9 P 5 8 0 0 0 0 1

ISBN/EAN/ISBN-13

Credit Card Numbers

German Bank Note

Transmission Errors

$$\underline{728166153146}$$

$$728166753146$$

$$721866153146$$

$$721866113546$$

$$728199153146$$

Common Error Patterns

Error Type	**Form**	**Relative Frequency**
single digit error	$a \to b$	79.1%
transposition of adjacent digits	$ab \to ba$	10.2%
jump transposition	$abc \to cba$	0.8%
twin error	$aa \to bb$	0.5%
phonetic error*	$a0 \leftrightarrow 1a$	0.5%
jump twin error	$aca \to bcb$	0.3%

*For $a = 2, 3, 4, 5, 6, 7, 8, 9$.

Modulo Arithmetic

Let x and n be integers with $n > 0$. The remainder r obtained when x is divided by n is denoted by **x (mod n)** where $0 \leq r \leq n - 1$.

Let n be a positive integer. Given two integers x and y, then **x is congruent to y modulo n**, denoted **x $\equiv$ y (mod n)**, if x and y have the same remainder when divided by n.

US Postal Money Order

General Form: $a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9 a_{10} a_{11}$

- Document Number: $a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9 a_{10}$
- Check Digit: a_{11}

$$a_{11} = (a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10}) \pmod 9$$

Valid Number: 67021200988

$$\begin{aligned}(6 + 7 + 0 + 2 + 1 + 2 + 0 + 0 + 9 + 8) \pmod 9 &= 35 \pmod 9 \\ &= 8\end{aligned}$$

Detection Rate

$$dr = \frac{\text{number of ways an error is detected}}{\text{number of ways to make an error}}$$

Single Digit Errors ($a \to b$): 10 choices for a and 9 choices for b resulting in 90 possible ways.

Transposition of Adjacent Digit Errors ($ab \to ba$): 10 choices for a and 9 choices for b resulting in 90 possible ways.

Detection Rate - USPMO

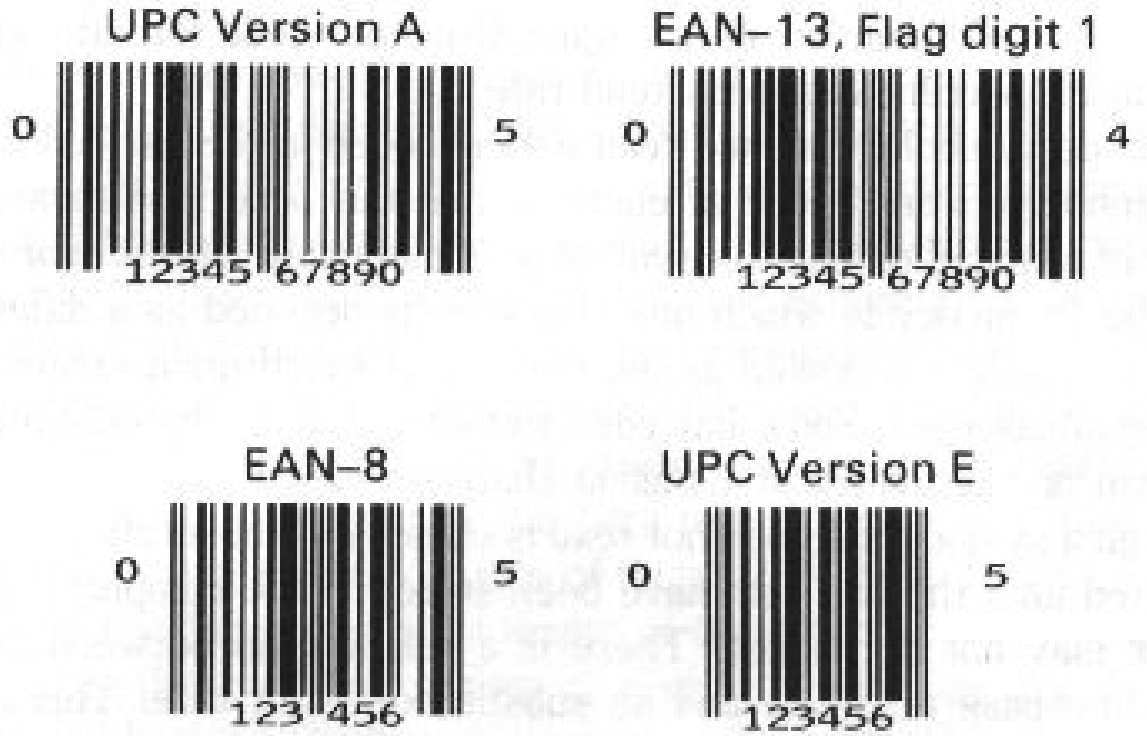
Single Digit Errors:

$$dr = \frac{88}{90} = 98\%$$

Transposition of Adjacent Digit Errors:

$$dr = \frac{0}{90} = 0\%$$

UPC and EAN


UPC Version A
0
5
12345 67890
EAN–13, Flag digit 1
0
4
12345 67890
EAN–8
0
5
123 456
UPC Version E
0
5
123456

UPC Version A

General Form: $a_1 - a_2a_3a_4a_5a_6 - a_7a_8a_9a_{10}a_{11} - a_{12}$

- a_1 - number system character
- $a_2a_3a_4a_5a_6$ - company number
- $a_7a_8a_9a_{10}a_{11}$ - product number
- a_{12} - check digit

$$3a_1 + a_2 + 3a_3 + a_4 + 3a_5 + a_6 + 3a_7 + a_8 + 3a_9 + a_{10} + 3a_{11} + a_{12} \equiv 0 \pmod{10}$$

No. System Char: 0 - General Groceries; 2 - Meat, Produce, Weight Items; 3 - Drugs and Health Products; 4 - In-Store Items; 5 - Coupons; 6,7 - Other Items;

UPC Version A

ID Number: 5-02003-91562 UPC: 5-02003-91562-C

$$3{\cdot}5 + 0 + 3{\cdot}2 + 0 + 3{\cdot}0 + 3 + 3{\cdot}9 + 1 + 3{\cdot}5 + 6 + 3{\cdot}2 + C = 0 \pmod{10}$$
$$15 + 0 + 6 + 0 + 0 + 3 + 27 + 1 + 15 + 6 + 6 + C = 0 \pmod{10}$$
$$79 + C = 0 \pmod{10}$$

Thus $C = 1$ and the UPC is 5-02003-91562-1.

UPC Version A - Single Digit Errors

The UPC Check Digit Scheme catches all single digit errors.

$$\ldots a \ldots \rightarrow \ldots b \ldots$$

$$c + 3a = 0 \pmod{10} \qquad \& \qquad c + 3b = 0 \pmod{10}$$

$$\begin{aligned}
(c + 3a) - (c + 3b) &= 0 \pmod{10} \\
3a - 3b &= 0 \pmod{10} \\
3(a - b) &= 0 \pmod{10} \\
a - b &= 0 \pmod{10} \\
a &= b
\end{aligned}$$

UPC Version A - Transposition of Adjacent Digit Errors

The UPC Check Digit Scheme does not catch all transposition of adjacent digit errors.

$$\ldots ab \ldots \rightarrow \ldots ba \ldots$$

$$c + 3a + b = 0 \pmod{10} \qquad \& \qquad c + 3b + a = 0 \pmod{10}$$

$$\begin{aligned} (c + 3a + b) - (c + 3b + a) &= 0 \pmod{10} \\ 2a - 2b &= 0 \pmod{10} \\ 2(a - b) &= 0 \pmod{10} \end{aligned}$$

Undetected when $|a - b| = 5$.

UPC Version A - Detection Rates

Single Digit Errors:

$$dr = \frac{90}{90} = 100\%$$

Transposition of Adjacent Digit Errors:

$$dr = \frac{80}{90} = 89\%$$

Weighted Sums - Mod 10

$$\sum_{i=1}^{n} w_i a_i = 0 \pmod{10}$$

$$w_1 a_1 + w_2 a_2 + \cdots + w_n a_n = 0 \pmod{10}$$

w_i - weights

a_i - digits in the identification number

catch all single digits errors	w_i relatively prime to 10
catch all transposition of adjacent digit errors	$w_{i+1} - w_i$ relatively prime to 10

Weighted Sums in General

$$\sum_{i=1}^{n} w_i a_i = 0 \pmod{M}$$

$$w_1 a_1 + w_2 a_2 + \cdots + w_n a_n = 0 \pmod{M}$$

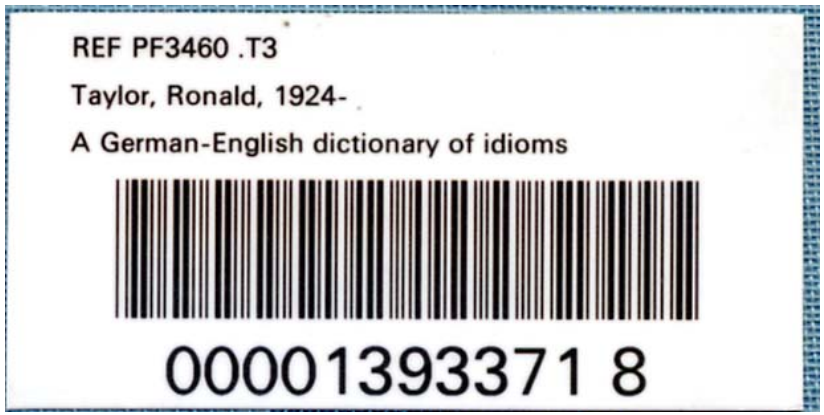
w_i - weights

a_i - digits in the identification number

catch all single digits errors	w_i relatively prime to M
catch all transposition of adjacent digit errors	$w_{i+1} - w_i$ relatively prime to M

IBM Scheme


BRITISH AIRWAYS
4417 1234 5678 9112
CARDMEMBER SINCE 1999
GOOD THRU 00/00
D. WALDEN
EXECUTIVE CLUB # 12345678
VISA
SIGNATURE


REF PF3460 .T3
Taylor, Ronald, 1924-
A German-English dictionary of idioms
00001393371 8

IBM Scheme

$$\sigma(a) = \begin{cases} 2a & \text{if } 0 \leq a \leq 4 \\ 2a - 9 & \text{if } 5 \leq a \leq 9 \end{cases} \qquad = 2a + \left\lfloor \frac{2a}{10} \right\rfloor \pmod{10}$$

$$\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 0 & 2 & 4 & 6 & 8 & 1 & 3 & 5 & 7 & 9 \end{pmatrix}$$

General Form : $a_1 a_2 \cdots a_{n-1} a_n$

n-even:

$$\sigma(a_1) + a_2 + \sigma(a_3) + a_4 + \cdots + \sigma(a_{n-1}) + a_n = 0 \pmod{10}$$

n-odd:

$$a_1 + \sigma(a_2) + a_3 + \sigma(a_4) + \cdots + \sigma(a_{n-1}) + a_n = 0 \pmod{10}$$

IBM Scheme

$$\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 0 & 2 & 4 & 6 & 8 & 1 & 3 & 5 & 7 & 9 \end{pmatrix}$$

Specific Number: 1324136 9

$$\begin{aligned} \sigma(1) + 3 + \sigma(2) + 4 + \sigma(1) + 3 + \sigma(6) + 9 \pmod{10} & \\ = 2 + 3 + 4 + 4 + 2 + 3 + 3 + 9 \pmod{10} & \\ = 30 \pmod{10} & \\ = 0 & \end{aligned}$$

IBM Scheme - Single Digit Errors

The IBM Scheme catches all single digit errors.

$$\ldots a \ldots \rightarrow \ldots b \ldots$$

$$c + \sigma(a) = 0 \pmod{10} \qquad \& \qquad c + \sigma(b) = 0 \pmod{10}$$

$$\begin{aligned}
(c + \sigma(a)) - (c + \sigma(b)) &= 0 \pmod{10} \\
\sigma(a) - \sigma(b) &= 0 \pmod{10} \\
\sigma(a) &= \sigma(b) \pmod{10} \\
\sigma(a) &= \sigma(b) \\
a &= b
\end{aligned}$$

IBM Scheme - Transposition of Adjacent Digit Errors

The IBM Scheme does not catch all transposition of adjacent digit errors.

$$\ldots ab \ldots \rightarrow \ldots ba \ldots$$

$$c+\sigma(a)+b = 0 \pmod{10} \qquad \& \qquad c+\sigma(b)+a = 0 \pmod{10}$$

$$\begin{aligned} (c+\sigma(a)+b)-(c+\sigma(b)+a) &= 0 \pmod{10} \\ \sigma(a)-a-\sigma(b)+b &= 0 \pmod{10} \\ \sigma(a)-a &= \sigma(b)-b \pmod{10} \end{aligned}$$

Undetected when $a = 0, b = 9$ or $a = 9, b = 0$.

IBM Scheme - Transposition of Adjacent Digit Errors

$0 \leq a \leq 4$	$5 \leq b \leq 9$
$\sigma(a) - a \pmod{10}$	$\sigma(b) - b \pmod{10}$
$2a - a \pmod{10}$	$2b - 9 - b \pmod{10}$
$a \pmod{10}$	$b - 9 \pmod{10}$
	$b + 1 \pmod{10}$
0	9

IBM Scheme - Transposition of Adjacent Digit Errors

$$\sigma(0) - 0 = 0 - 0 = 0$$
$$\sigma(1) - 1 = 2 - 1 = 1$$
$$\sigma(2) - 2 = 4 - 2 = 2$$
$$\sigma(3) - 3 = 6 - 3 = 3$$
$$\sigma(4) - 4 = 8 - 4 = 4$$
$$\sigma(5) - 5 = 1 - 5 = 6$$
$$\sigma(6) - 6 = 3 - 6 = 7$$
$$\sigma(7) - 7 = 5 - 7 = 8$$
$$\sigma(8) - 8 = 7 - 8 = 9$$
$$\sigma(9) - 9 = 9 - 9 = 0$$

IBM Detection Rates

Single Digit Errors:

$$dr = \frac{90}{90} = 100\%$$

Transposition of Adjacent Digit Errors:

$$dr = \frac{88}{90} = 98\%$$

A Theorem by Gumm, 1985

Suppose an error detecting scheme with an even modulus detects all single digit errors. Then for every i and j there is a transposition error involving positions i and j that cannot be detected.

$$a_1 \cdots a_i \cdots a_j \cdots a_n \to a_1 \cdots a_j \cdots a_i \cdots a_n$$

Proof:

- Let $2m$ be the even modulus.
- For $a_1 a_2 \cdots a_n$, the CDS is

$$\sigma_1(a_1) + \sigma_2(a_2) + \cdots \sigma_n(a_n) = 0 \pmod{2m}$$

- To catch all single digit errors, each σ_i must be a permutation of $\mathbb{Z}_{2m} = \{0, 1, 2, \ldots, 2m-1\}$.

A Theorem by Gumm, 1985

- To detect all transposition errors involving positions i and j ($\ldots a \ldots b \ldots \to \ldots b \ldots a \ldots$, $a \neq b$), we must have

$$\sigma_i(a) + \sigma_j(b) \neq \sigma_i(b) + \sigma_j(a)$$

or

$$\sigma_j(b) - \sigma_i(b) \neq \sigma_j(a) - \sigma_i(a)$$

- In this case, the map $\sigma(x) = \sigma_j(x) - \sigma_i(x)$ is a permutation of $\mathbb{Z}_{2m}$.

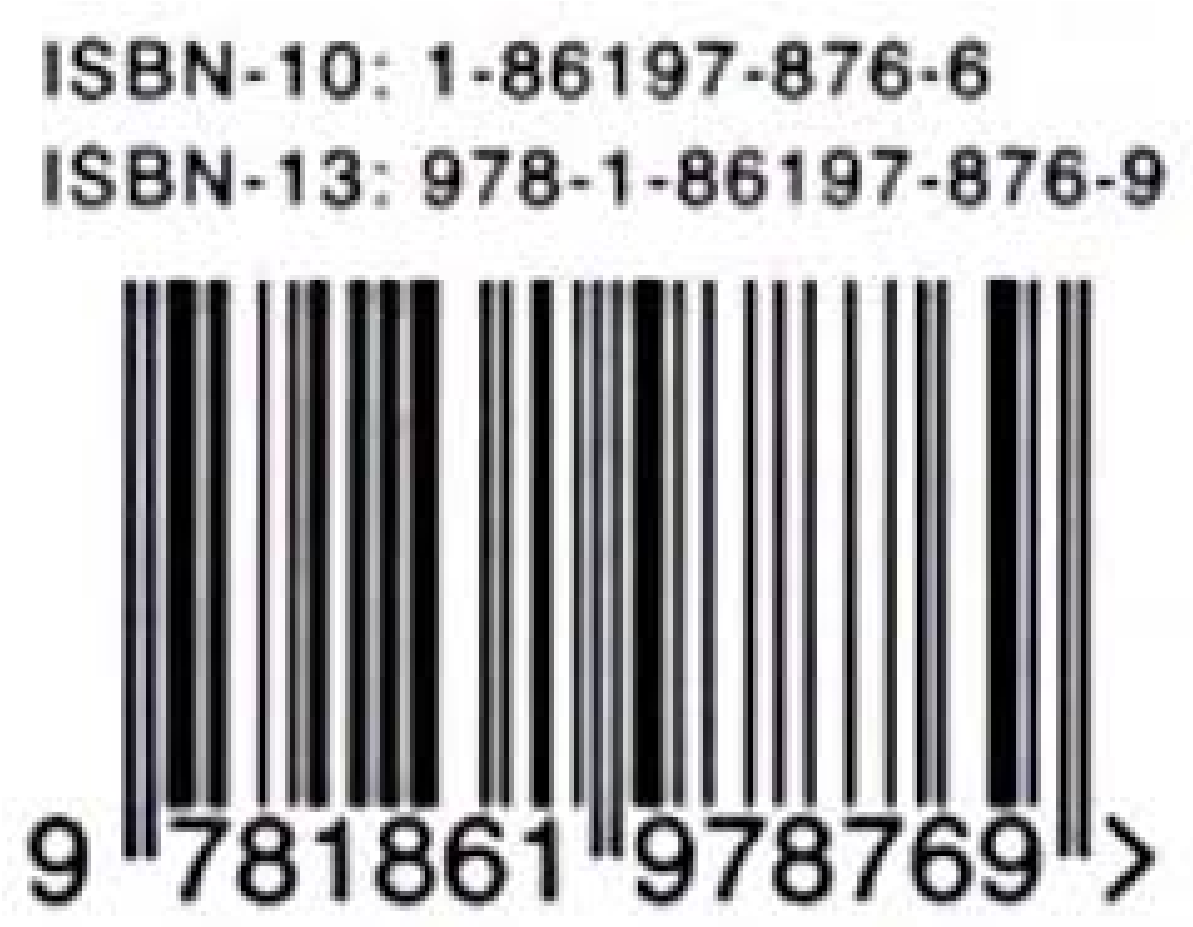
A Theorem by Gumm, 1985

- Summing the elements of $\mathbb{Z}_{2m}$ (mod $2m$) obtains

$$\begin{aligned} m &= m + 0 + (1 + 2m - 1) + (2 + 2m - 2) + \ + (m - 1 + m + 1) \\ &= \sum x \\ &= \sum \sigma(x) \\ &= \sum (\sigma_j(x) - \sigma_i(x)) \\ &= \sum \sigma_j(x) - \sum \sigma_i(x) \\ &= m - m \\ &= 0 \end{aligned}$$

This is a contradiction.

ISBN...ISBN-10...ISBN-13...EAN-13...EAN



ISBN-10

General Form: $a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9 a_{10}$

- $a1\ldots$: group/country number (0,1=English, 3=German, 9978=Ecuador)
- $a_i \ldots a_j$: publisher number
- $a_{j+1} \ldots a_9$: serial number
- a_{10}: check digit

$$10 \cdot a_1 + 9 \cdot a_2 + 8 \cdot a_3 + 7 \cdot a_4 + 6 \cdot a_5 + 5 \cdot a_6 + 4 \cdot a_7 + 3 \cdot a_8 + 2 \cdot a_9 + 1 \cdot a_{10} \equiv 0 \pmod{11}$$

If $a_{10} = 10$, the letter X is used.

ISBN-10

ISBN-10: 1-86197-876-6

$$10{\cdot}1 + 9{\cdot}8 + 8{\cdot}6 + 7{\cdot}1 + 6{\cdot}9 + 5{\cdot}7 + 4{\cdot}8 + 3{\cdot}7 + 2{\cdot}6 + 1{\cdot}6 \equiv 0 \pmod{11}$$

$$10 + 72 + 48 + 7 + 54 + 35 + 32 + 21 + 12 + 6 \equiv 0 \pmod{11}$$

$$297 = 17{\cdot}11 \equiv 0 \pmod{11}$$

ISBN-10 Detection Rates

Single Digit Errors:

$$dr = \frac{90}{90} = 100\%$$

Transposition of Adjacent Digit Errors:

$$dr = \frac{90}{90} = 100\%$$

But why don't we like this scheme?

German Banks

German banks use a mod 11 weighted scheme that does catch all of the errors from the first table and any transposition error when $n \leq 10$.

$$2a_1 + 2^2a_2 + 2^3a_3 + \cdots + 2^na_n = 0 \pmod{11}$$

Transposition involving position i and j where $1 \leq i < j \leq n$.

$$\ldots a \ldots b \ldots \rightarrow \ldots b \ldots a \ldots$$

$$c+2^ia+2^jb = 0 \pmod{11} \qquad \& \qquad c+2^ib+2^ja = 0 \pmod{11}$$

$$2^i(2^{j-i} - 1)b = 2^i(2^{j-i} - 1)a \pmod{11}$$

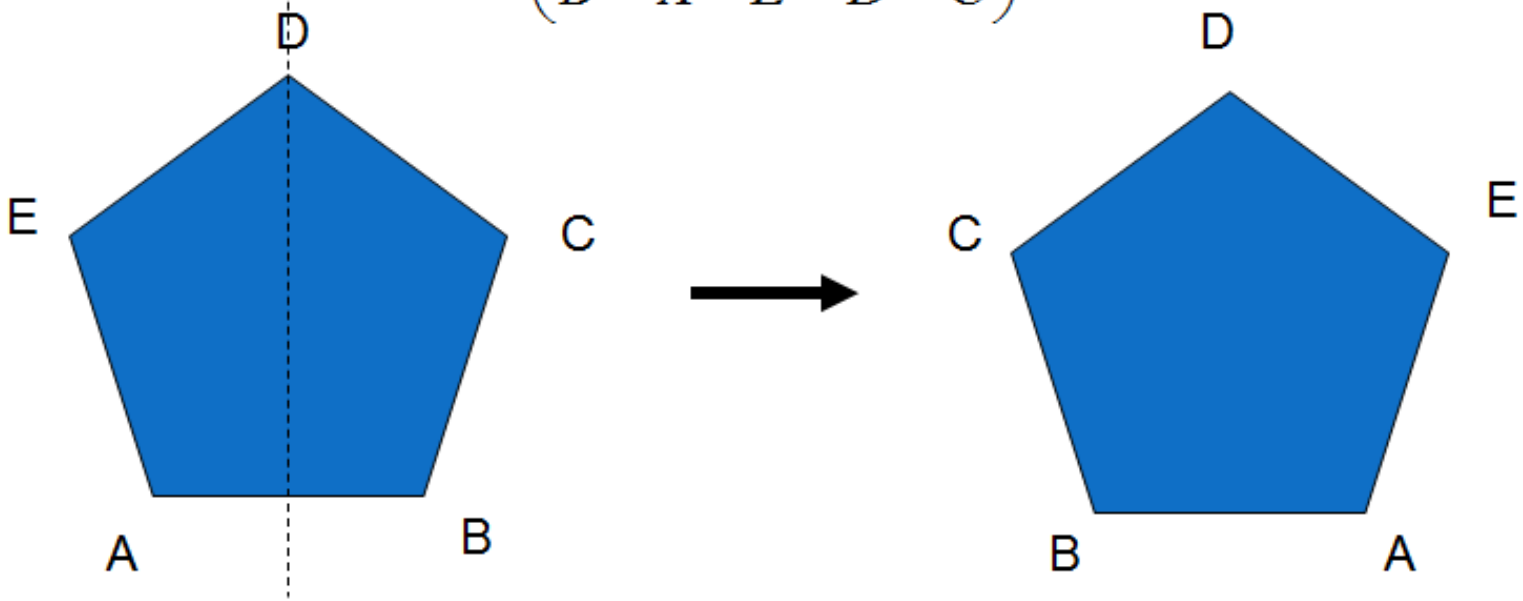
German Banks

$$2^1 - 1 = 1$$
$$2^2 - 1 = 3$$
$$2^3 - 1 = 7$$
$$2^4 - 1 = 15 = 3 \cdot 5$$
$$2^5 - 1 = 31$$
$$2^6 - 1 = 63 = 3^2 \cdot 7$$
$$2^7 - 1 = 127$$
$$2^8 - 1 = 255 = 3 \cdot 5 \cdot 17$$
$$2^9 - 1 = 511 = 7 \cdot 73$$
$$2^{10} - 1 = 1023 = 3 \cdot 11 \cdot 31$$

Symmetries of a Regular Pentagon

Reflection

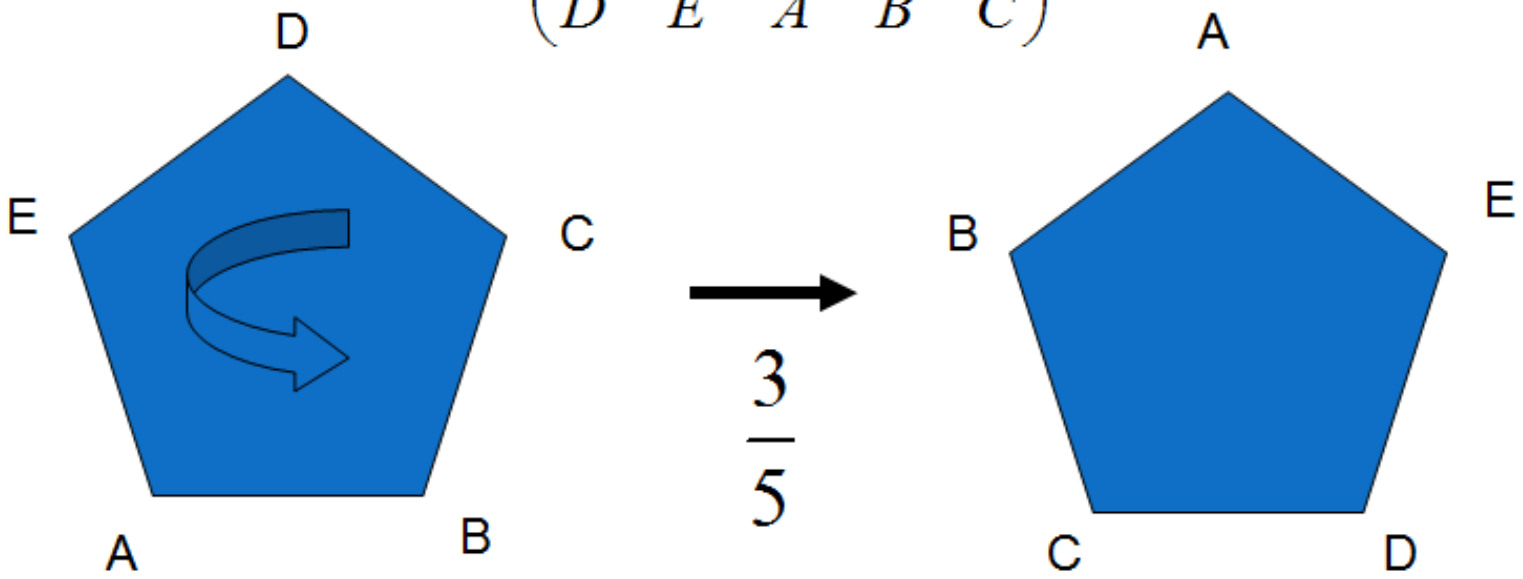
$$\begin{pmatrix} A & B & C & D & E \\ B & A & E & D & C \end{pmatrix}$$



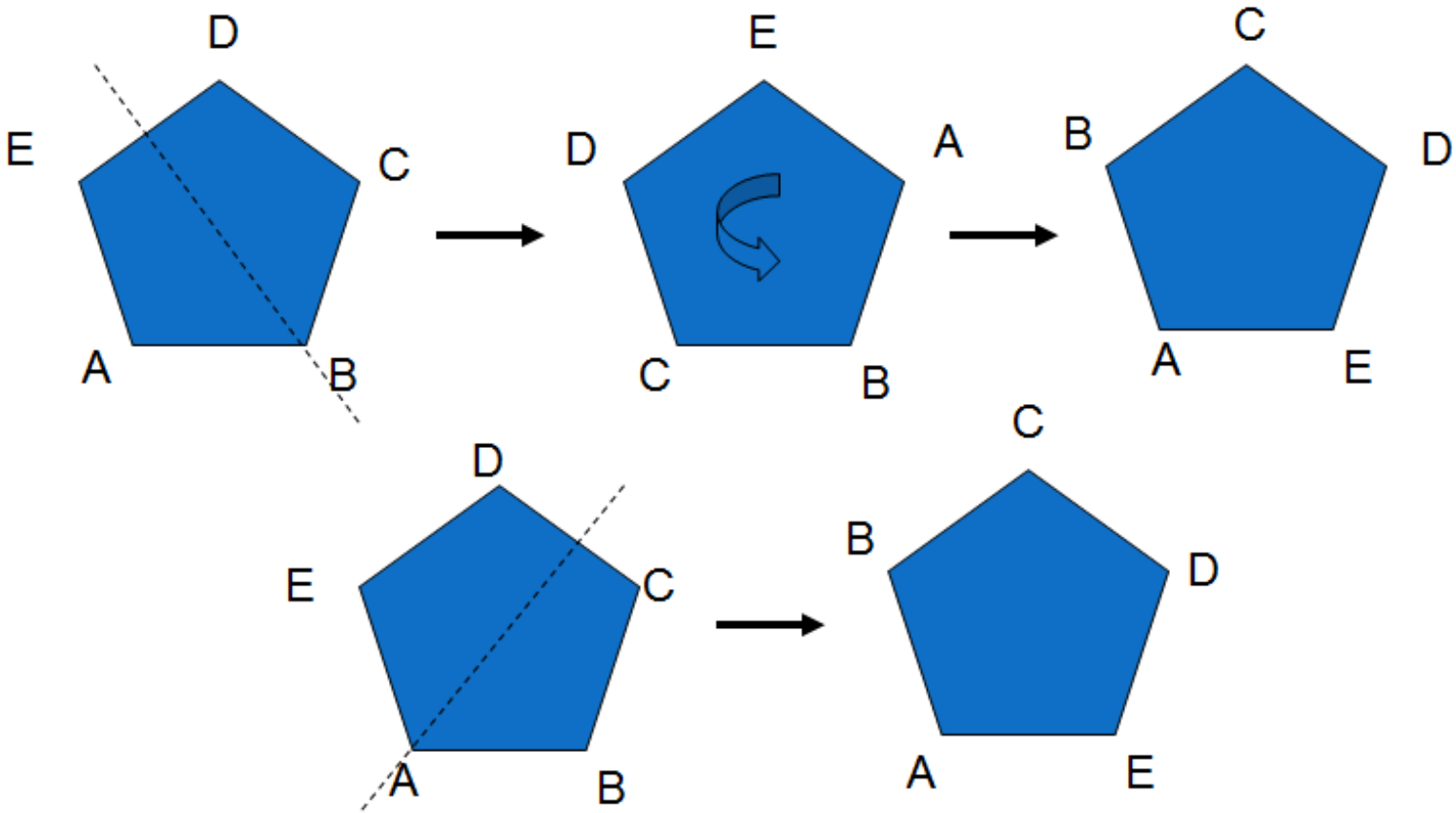
Symmetries of the Regular Pentagon

Rotations

$$\begin{pmatrix} A & B & C & D & E \\ D & E & A & B & C \end{pmatrix}$$



Symmetries of a Regular Pentagon



Symmetries of a Regular Pentagon

$$0 = \begin{pmatrix} A & B & C & D & E \\ A & B & C & D & E \end{pmatrix} \quad 1 = \begin{pmatrix} A & B & C & D & E \\ B & C & D & E & A \end{pmatrix}$$

$$2 = \begin{pmatrix} A & B & C & D & E \\ C & D & E & A & B \end{pmatrix} \quad 3 = \begin{pmatrix} A & B & C & D & E \\ D & E & A & B & C \end{pmatrix}$$

$$4 = \begin{pmatrix} A & B & C & D & E \\ E & A & B & C & D \end{pmatrix} \quad 5 = \begin{pmatrix} A & B & C & D & E \\ A & E & D & C & B \end{pmatrix}$$

$$6 = \begin{pmatrix} A & B & C & D & E \\ E & D & C & B & A \end{pmatrix} \quad 7 = \begin{pmatrix} A & B & C & D & E \\ D & C & B & A & E \end{pmatrix}$$

$$8 = \begin{pmatrix} A & B & C & D & E \\ C & B & A & E & D \end{pmatrix} \quad 9 = \begin{pmatrix} A & B & C & D & E \\ B & A & E & D & C \end{pmatrix}$$

The Multiplication Table for D_5

$8 * 3 = 5$ $\quad$ $3 * 8 = 6$ $\quad$ The operation is not commutative!

$*$	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	0	6	7	8	9	5
2	2	3	4	0	1	7	8	9	5	6
3	3	4	0	1	2	8	9	5	6	7
4	4	0	1	2	3	9	5	6	7	8
5	5	9	8	7	6	0	4	3	2	1
6	6	5	9	8	7	1	0	4	3	2
7	7	6	5	9	8	2	1	0	4	3
8	8	7	6	5	9	3	2	1	0	4
9	9	8	7	6	5	4	3	2	1	0

The Verhoeff Scheme

- Number: $a_1 a_2 \cdots a_n$
- Permutation: $\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 0 & 4 & 3 & 2 & 1 & 6 & 7 & 8 & 9 & 5 \end{pmatrix}$ not unique

  Any permutation works as long as

$$\sigma(a) * b \neq \sigma(b) * a \qquad \text{for } a \neq b.$$

- $*$ is the group operation from D_5.

$$\sigma^{n-1}(a_1) * \sigma^{n-2}(a_2) * \sigma^{n-3}(a_3) * \cdots * \sigma(a_{n-1}) * a_n = 0$$

The Verhoeff Scheme

Note: Since σ is a permutation, for each $i \in \mathbb{N}$, σ^i is also a permutation.

$$\sigma^3 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 0 & 4 & 3 & 2 & 1 & 6 & 7 & 8 & 9 & 5 \end{pmatrix} \circ$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 0 & 4 & 3 & 2 & 1 & 6 & 7 & 8 & 9 & 5 \end{pmatrix} \circ \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 0 & 4 & 3 & 2 & 1 & 6 & 7 & 8 & 9 & 5 \end{pmatrix}$$

$$\sigma^3 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 0 & 4 & 3 & 2 & 1 & 8 & 9 & 5 & 6 & 7 \end{pmatrix}$$

The Verhoeff Scheme - Single Digit Errors

Catches all single digit errors.

$$\ldots a \ldots \rightarrow \ldots b \ldots$$

$$c_1 * \sigma^i(a) * c_2 = 0 \qquad \& \qquad c_1 * \sigma^i(b) * c_2 = 0$$

$$\begin{aligned} c_1 * \sigma^i(a) * c_2 &= c_1 * \sigma^i(b) * c_2 \\ \sigma^i(a) * c_2 &= \sigma^i(b) * c_2 \\ \sigma^i(a) &= \sigma^i(b) \\ a &= b \end{aligned}$$

The Verhoeff Scheme - Transposition of Adj. Digit Errors

Catches all transposition of adjacent digit errors.

$$\ldots ab \ldots \rightarrow \ldots ba \ldots$$

$$c_1 * \sigma^{i+1}(a) * \sigma^i(b) * c_2 = 0 \qquad \& \qquad c_1 * \sigma^{i+1}(b) * \sigma^i(a) * c_2 = 0$$

$$\begin{aligned} c_1 * \sigma^{i+1}(a) * \sigma^i(b) * c_2 &= c_1 * \sigma^{i+1}(b) * \sigma^i(a) * c_2 \\ \sigma^{i+1}(a) * \sigma^i(b) * c_2 &= \sigma^{i+1}(b) * \sigma^i(a) * c_2 \\ \sigma^{i+1}(a) * \sigma^i(b) &= \sigma^{i+1}(b) * \sigma^i(a) \\ \sigma(\sigma^i(a)) * \sigma^i(b) &= \sigma(\sigma^i(b)) * \sigma^i(a) \end{aligned}$$

The Verhoeff Scheme

$$\sigma(\sigma^i(a)) * \sigma^i(b) = \sigma(\sigma^i(b)) * \sigma^i(a)$$

Since $a \neq b$, it follows that $\sigma^i(a) \neq \sigma^i(b)$. Let $c = \sigma^i(a)$ and $d = \sigma^i(b)$. Then

$$\sigma(\sigma^i(a)) * \sigma^i(b) = \sigma(\sigma^i(b)) * \sigma^i(a)$$
$$\sigma(c) * d = \sigma(d) * c$$

This is a contradiction.

The German Bundesbank Scheme

The German Bundesbank Scheme

- Number: $a_1 a_2 \cdots a_{11}$
- Permutation: $\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 5 & 7 & 6 & 2 & 8 & 3 & 0 & 9 & 4 \end{pmatrix}$
-

A	D	G	K	L	N	S	U	Y	Z
0	1	2	3	4	5	6	7	8	9

- $*$ is the group operation from D_5.

$$\sigma(a_1) * \sigma^2(a_2) * \sigma^3(a_3) * \cdots * \sigma^{10}(a_{10}) * a_{11} = 0$$

The German Bundesbank Scheme

$$\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 5 & 7 & 6 & 2 & 8 & 3 & 0 & 9 & 4 \end{pmatrix}$$

$\text{DL0998939U1} \rightarrow 14099893971$

$$\sigma(1) * \sigma^2(4) * \sigma^3(0) * \sigma^4(9) * \sigma^5(9) * \sigma^6(8) *$$
$$\sigma^7(9) * \sigma^8(3) * \sigma^9(9) * \sigma^{10}(7) * 1 = 0$$
$$5 * 7 * 8 * 0 * 1 * 1 * 8 * 3 * 4 * 1 * 1 = 0$$
$$0 = 0$$

German Bundesbank Scheme

This scheme has one major problem, what is it?

Euro

An Error Correcting Scheme

Number: $a_1 a_2 \cdots a_8 a_9 a_{10}$ with a_9, a_{10} check digits.

$$a_1 + a_2 + \ldots + a_8 + a_9 + a_{10} = 0 \pmod{11}$$

$$a_1 + 2a_2 + \ldots + 8a_8 + 9a_9 + 10a_{10} = 0 \pmod{11}$$

An Error Correcting Scheme

$62150334a_9a_{10}$

$$a_1 + a_2 + \ldots + a_8 + a_9 + a_{10} = 0 \pmod{11}$$
$$6 + 2 + 1 + 5 + 0 + 3 + 3 + 4 + a_9 + a_{10} = 0 \pmod{11}$$
$$24 + a_9 + a_{10} = 0 \pmod{11}$$
$$2 + a_9 + a_{10} = 0 \pmod{11}$$

$$a_1 + 2a_2 + \ldots + 8a_8 + 9a_9 + 10a_{10} = 0 \pmod{11}$$
$$6 + 2\cdot 2 + 3\cdot 1 + 4\cdot 5 + 5\cdot 0 + 6\cdot 3 + 7\cdot 3 + 8\cdot 4 + 9\cdot a_9 + 10\cdot a_{10} = 0 \pmod{11}$$
$$104 + 9\cdot a_9 + 10\cdot a_{10} = 0 \pmod{11}$$
$$5 + 9\cdot a_9 + 10\cdot a_{10} = 0 \pmod{11}$$

An Error Correcting Scheme

$$6215033472 \rightarrow 6218033472$$

$$\begin{aligned} a_1 + a_2 + \ldots + a_8 + a_9 + a_{10} &\pmod{11} \\ 6 + 2 + 1 + 8 + 0 + 3 + 3 + 4 + 7 + 2 &\pmod{11} \\ 36 &\pmod{11} = 3 \end{aligned}$$

$$\begin{aligned} a_1 + 2a_2 + \ldots + 8a_8 + 9a_9 + 10a_{10} &= 3i \pmod{11} \\ 6 + 2\cdot 2 + 3\cdot 1 + 4\cdot 8 + 5\cdot 0 + 6\cdot 3 + 7\cdot 3 + 8\cdot 4 + 9\cdot 7 + 10\cdot 2 &= 3i \pmod{11} \\ 199 &= 3i \pmod{11} \\ 1 &= 3i \pmod{11} \\ i &= 4 \end{aligned}$$

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