Two Easy Theorems From Financial Folklore

(1) PORTFOLIO REBALANCING INCREASES VALUE

Bob Cotton April 19, 2021

Portfolio Rebalancing

Every investment guide recommends diversification of holdings, along with periodic rebalancing, to produce growth with reduced risk.

Rebalancing increases the size of the inventory when there is substantial volatility in the goods' relative prices.

Is there a name for this principle? I have not found it. Let's try to state and prove it.

Problem Statement

- An inventory contains n types of goods.
- The manager tries to maintain the value of type number k at a fraction w_k of the total market value, where $\sum_{k=1}^{n} w_k = 1$.
- Find a measure of inventory quantity and a strategy so that the quantity, so measured, increases over time.

Means in General

- Arithmetic mean: $\frac{(a+b+c)}{3} = \frac{1}{3} \cdot a + \frac{1}{3} \cdot b + \frac{1}{3} \cdot c$
- Geometric mean: $\sqrt[3]{a \cdot b \cdot c} = \sqrt[3]{a} \cdot \sqrt[3]{b} \cdot \sqrt[3]{c} = a^{1/3} \cdot b^{1/3} \cdot c^{1/3}$
- Weighted geometric mean: $a^{w_1} \cdot b^{w_2} \cdot c^{w_3}$ where $w_1 + w_2 + w_3 = 1$

The Weighted Geometric Mean

- Let v_k be the total market value of the portfolio goods of type k.
- Let $\sum_{k=1}^{n} w_k = 1$
- The weighted mean value of the portfolio is

$$V = \prod_{k=1}^{n} v_k^{w_k}$$

- We will show that (a) rebalancing maximizes V. This in turn (b) increases Q, our measure of portfolio quantity.

Symbols for Price and Quantity Means

Define also

$p_k =$ price per unit of good number k

$q_k =$ number of units of good number k in inventory

So that now $v_k = p_k \cdot q_k$

Corresponding geometric means for the p's and q's are

$$P = \prod_{k=1}^{n} p_k^{w_k} \text{ and } Q = \prod_{k=1}^{n} q_k^{w_k}$$

Proof of the Easier Theorem (b)

Maximizing V increases Q

$$V = \prod_{k=1}^{n} v_k^{w_k} = \prod_{k=1}^{n} (p_k q_k)^{w_k} = \prod_{k=1}^{n} p_k^{w_k} q_k^{w_k} = \prod_{k=1}^{n} p_k^{w_k} \prod_{k=1}^{n} q_k^{w_k} = P \cdot Q$$

Since rebalancing is fairly rapid, P may be assumed constant during this process. Therefore, an increase in V to its current maximum must result in an increase in Q.

Inventory Shares vs. Equity Shares

- The quantity Q is a measure of *inventory shares.*
- A proportionality factor may be needed for Q to serve this purpose convincingly.
- Shares sold to owners of the inventory we will call *equity shares,* numbering N*.*
- Even if these two are equal initially, rebalancing will cause them to diverge over time.

Value Preservation

The Strategy

- An exchange of one good for another, or an exchange of one good for equity shares, can be used to affect a rebalancing of the portfolio.

- To preserve the value of each share, the administrator may follow the rule that no transaction may decrease the ratio of total inventory shares Q to total equity shares N. We may call deals that increase this ratio *Q-profitable deals.*

Portfolio Market Value

- V is a mean value, with little intuitive meaning.
- The portfolio market value is

$$M = \sum_{k=1}^{n} v_k = \sum_{k=1}^{n} p_k q_k$$

- Rebalancing deals may not be profitable in terms of portfolio market value, due to transaction costs.
- Taxes also greatly affect profitability and Q-profitability.

Theorem (b)

Restatement

- Suppose weights $w_1, ..., w_n$ are fixed once for all with $\sum_{k=1}^{n} w_k = 1$
- The weighted geometric mean Q provides a measure of portfolio size.
- The strategy of using only Q-profitable deals in rebalancing insures that equity share value so measured cannot decrease over time.
- Whenever the relative prices of portfolio goods change significantly, Q/N may be increased by rebalancing.

Towards Theorem (a)

Rebalancing increases the geometric mean V

- This follows quickly from an economic principle known as the *equi-marginal principle* or *Gossen's Second Law*.
- Let $f(x_1, \ldots, x_n)$ be a positive function of positive variables, and *f* has partial derivatives.
- We wish to find the maximum of f subject to the constraint that

$$\sum_{k-1}^{n} x_k = K$$

- Claim: at such a maximum, $\frac{\partial f}{\partial x_k} = \frac{\partial f}{\partial x_j}$ for all k and j from 1,…,n.

Proof of theorem (a)

- Apply Gossen's Second Law with f = V and $x_k = v_k$ and K = 1.

Since $v_k^{w_k} = e^{w_k \cdot ln(v_k)}$

$$\frac{\partial V}{\partial v_i} = \frac{\partial}{\partial v_i} \prod_{k=1}^{n} e^{w_k \cdot ln(v_k)} = \frac{\partial}{\partial v_i} e^{\sum_{k=1}^{n} w_k \cdot ln(v_k)} = V \frac{\partial}{\partial v_i}(w_i \cdot ln(v_i)) = V \frac{w_i}{v_i}$$

Is Q adequate to measure quantity?

What if relative prices returned to their original values?

- If relative prices return to their original values and Q has increased, then every q_k has increased. We show the original values without primes, final values are primed.
- Given that the following three quantities are independent of k:

$\frac{p_k q_k}{w_k}, \quad \frac{p'_k q'_k}{w_k}, \quad \frac{p'_k}{p_k}$ we want that $\frac{q'_k}{q_k}$ is also.

$$\frac{q'_k}{q_k} = \frac{\frac{p'_k q'_k}{w_k}}{\frac{p_k q_k}{w_k}} \cdot \frac{p_k}{p'_k}$$

It remains to relate this to Q'/Q.

(Proof continued)

Let $K = \dfrac{q'_k}{q_k}$

$$Q' = \prod_{k=1}^{n} (q'_k)^{w_k} = \prod_{k=1}^{n} (Kq_k)^{w_k} = \prod_{k=1}^{n} K^{w_k} \prod_{k=1}^{n} q_k^{w_k} = K \cdot Q$$

Gossen's Laws, Generalized

From differentiable to convex functions

1. The law of diminishing marginal utility.

2. The law of equal marginal utility across all goods and services.

A flnite set of points in the plane, think push-pins in a bulletin board, when surrounded by an elastic band form a convex polygon, called a *convex hull*. The upper part of this hull shows diminishing slope.

The notion of tangent to a curve can be generalized to lines that meet a convex curve at one or more points, but which are never below the curve. With this, the second law can be restated for convex functions.

A Currency Exchange Index

Using M-Matrices

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Currency Exchanges

- Currency exchanges are essential to foreign trade
- They could also provide a way to trade a type of debt currency based on labor IOUs.
- We show a simple way to compute a scale of relative prices, using M-matrices and a history of recent exchanges.
- Such a scale provides an aid valuable to traders.

Given *n* distinct currencies, there are $n(n-1)/2$ pairs to trade.
Each pair has an exchange rate.
The rates may not be mutually consistent.
That is, $R(a,b) \times R(b,c) = R(a,c)$ may fail.
Then the exchange may profit from arbitrage.

Questions:

- Can we eliminate the middleman?
- Can we build a value scale for the consumer to use?
- Consumers would use multiple currencies.

Consequences

Why It Matters

- A positive answer would permit a type of debt currency.
- Each worker issues IOUs for promises of hours to be worked.
- Freely trade these with others. Store the trade details in a common database.
- We will compute a single price index list that expresses each currency’s value relative to the rest, permitting traders to make informed deals.
- In other uses, it could eliminate the need for arbitrage and its added costs.

Elementary Matrix A

0	0	0	0	0
0	5	0	-3	0
0	0	0	0	0
0	-5	0	3	0
0	0	0	0	0

Elementary Matrix Meaning

- Assign row/column numbers 1, … , n to distinct currencies.
- Each trade is entered twice.
- Enter amounts paid on the diagonal, as positive numbers.
- Enter amounts received by the payer in the same row and in the appropriate column, as negative numbers.
- Note that the sum of each column is zero.
- All diagonal entries are positive, off-diagonal entries are negative.

Vector in Nullspace Gives Relative Values

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -5 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 0 \\ 5 \\ 0 \end{bmatrix}$$

Build a Combined Matrix

Using the database of exchanges, sum the elementary matrices.

- The columns still add to zero, so the matrix is singular.
- Each row k contains sums from all exchanges that involve currency k.
- The total amount of currency k paid is A[k,k], and the currencies received in exchange are negative numbers off the diagonal in row k. The product of any row k with a solution column vector is zero. This says that the sums of the trades involving currency k are fair according to the solution prices.

Example:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 & 0 & 0 \\ 0 & 2 & -3 & 0 & 0 \\ -1 & -2 & 12 & -4 & -5 \\ 0 & 0 & -3 & 4 & 0 \\ 0 & 0 & -3 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1/2 \\ 1/3 \\ 1/4 \\ 1/5 \end{bmatrix}$$

What is Row Echelon Form?

- It is found by applying row operations to the matrix. These are

1. Multiply a row by a non-zero constant.
2. Exchange two rows.
3. Add a multiple of one row to another.

In the final row echelon form,

- The first non-zero element of each row is 1.
- In each column containing a leading one, the other entries are zero.

Each operation can be performed by multiplication from the left by a suitable matrix.

Row Echelon Form and Solution

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & -5 \\ 0 & 1 & 0 & 0 & -5/2 \\ 0 & 0 & 1 & 0 & -5/3 \\ 0 & 0 & 0 & 1 & -5/4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 5/2 \\ 5/3 \\ 5/4 \\ 1 \end{bmatrix}$$

- If matrix A has rank n-1 or 4 here, then the last column provides the negative of the first n-1 entries of a solution vector.
- This solution is a multiple of the one shown before.

General Case

When is the rank of A equal to n-1?

- The row echelon form of A has one zero row.
- Let B be the n-1 by n-1 principal submatrix, consisting of the first n-1 rows and columns of A.
- A has rank n-1 if and only if B is invertible (non-singular).

A Graphical Condition

- Think of the diagonal elements of A, representing currencies, as the nodes of a graph.
- Two nodes $A[i,i]$ and $A[k,k]$ are joined by an edge if $A[i,k] \neq 0$.
- We claim that B is invertible if and only if this graph is connected.
- “Connected” means here that the currencies cannot be separated onto two subsets in such a way that there are no exchanges between the currencies of the different subsets.
- This claim is not trivial to prove, but we will sketch some helpful ideas.

M-Matrix Inverse

If it exists, it's positive!

- Recall that an M-Matrix is positive on the diagonal, negative off the diagonal.
- Both our matrix A and principal submatrix B are M-matrices.

Let $B = D - E$ where both D and E are all positive. D is a diagonal matrix; E contains only off-diagonal elements.

Note that D^{-1} is the diagonal matrix whose elements are the reciprocals of those of D. Define $F = ED^{-1}$ so that

$$BD^{-1} = (D - E)D^{-1} = I - ED^{-1} = I - F$$

M-Matrix (cont.)

- An algebraic trick:

$$(I - F)(I + F + F^2 + \ldots + F^k) = I - F^{k+1}$$

- From this we see that if $F^{k+1} \to 0$ as $k \to \infty$ then $(I - F)^{-1}$ is the sum of an infinite series of positive matrices.
- This provides a strategy for proving the earlier claim: B is invertible if and only if a certain graph is connected. The connectedness of a graph allows us to prove convergence here.

Normalization

Making the solution vector unique

- So far our solution vector of relative prices is only unique up to a multiplicative factor. It would be convenient to normalize it in some way. Possibilites:

1. Make the coordinate average equal to 1.
2. Make the coordinate geometric mean equal to 1.
3. Make the coordinate median equal to 1.
4. Make the reciprocal coordinate average equal to 1.

(I like 4 best)