Unsubstantiated Facts about Binary Gates

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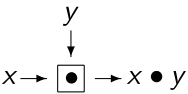
— *Poughkeepsie ACM Chapter* —

A **nand gate** is a small chip N with two $0,1$-inputs x and y and a $0,1$-output $x \bullet y$.



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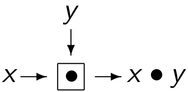

		y	
	$\bullet$	0	1
x	0	1	1
	1	1	0

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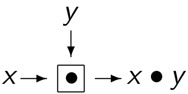

$\bullet$	$y=0$	$y=1$
$x=0$	1	1
$x=1$	1	0

If 1 is "TRUE" and 0 is "FALSE", then the $\wedge$ and $\vee$ gates are given by

$\wedge$	$y=0$	$y=1$
$x=0$	0	0
$x=1$	0	1

$\vee$	$y=0$	$y=1$
$x=0$	0	1
$x=1$	1	1

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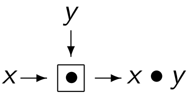

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	1	0	1

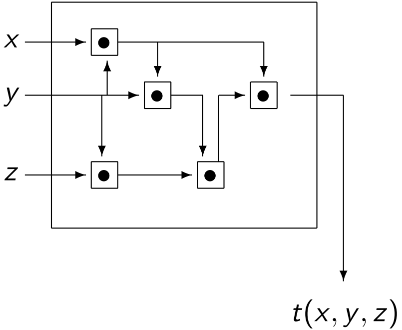
	$\vee$	$y = 0$	$y = 1$
x	0	0	1
	1	1	1

	$+$	$y = 0$	$y = 1$
x	0	0	1
	1	1	0

A **switching circuit** can be built from N gates.

N

$\bullet$	0	1
0	1	1
1	1	0



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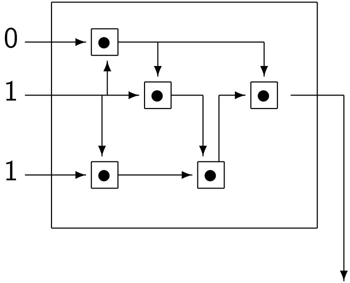

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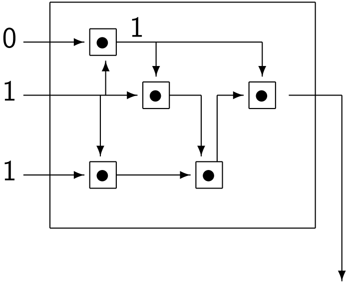

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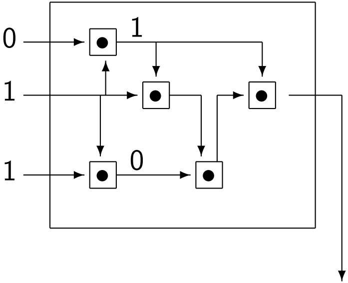

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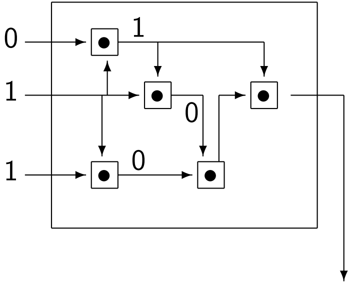

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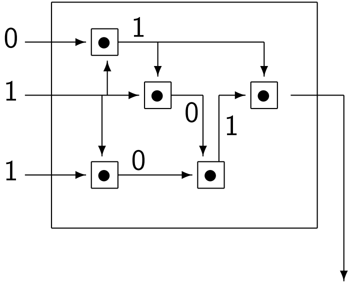

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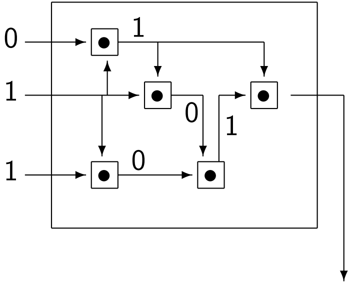


$t(0, 1, 1) = 0$

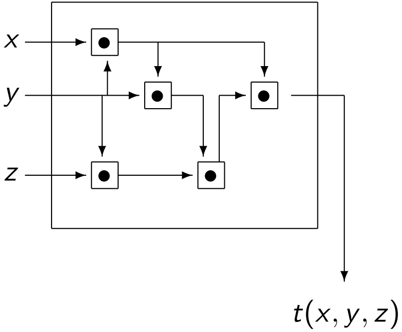
A **switching circuit** can be built from N gates.

N

•	0	1
0	1	1
1	1	0

Performance Table

x	y	z	$t(x,y,z)$
0	0	0	
0	0	1	
0	1	0	
0	1	1	
1	0	0	
1	0	1	
1	1	0	
1	1	1	



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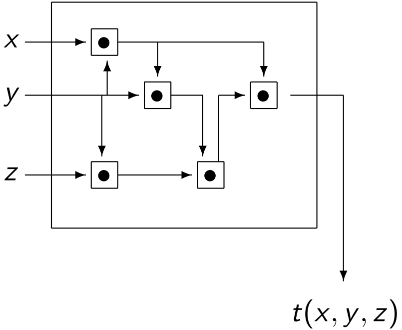
1	0	1	
1	1	0	
1	1	1	



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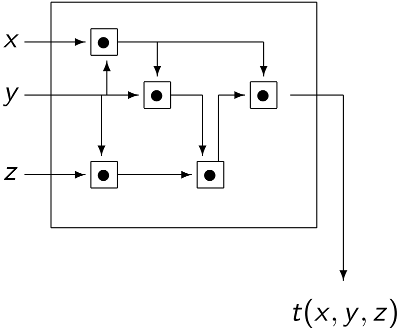
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1



An n-valued binary gate with $n = 5$.

G_1 $*$	0	1	2	3	4
0	1	1	0	2	0
1	4	2	0	3	3
2	1	0	3	2	4
3	0	4	3	4	2
4	2	0	1	4	0

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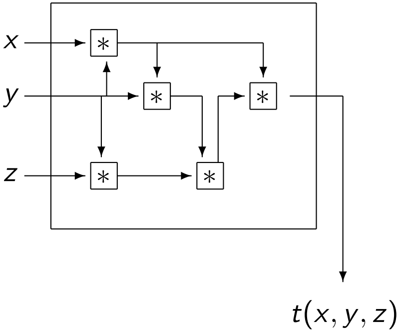
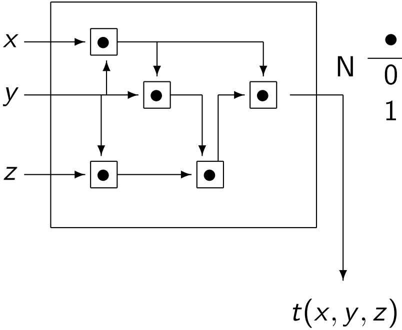
4	2	0	1	4	0

An n-valued switching circuit with $n = 5$.



Performance Table

x	y	z	$t(x,y,z)$
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

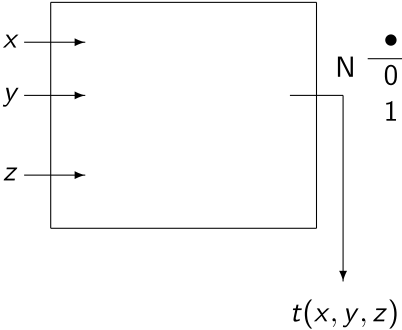


N

$\bullet$	0	1
0	1	1
1	1	0

Performance Table $\Leftarrow$ Switching Circuit

x	y	z	$t(x,y,z)$
0	0	0	
0	0	1	
0	1	0	
0	1	1	
1	0	0	
1	0	1	
1	1	0	
1	1	1	



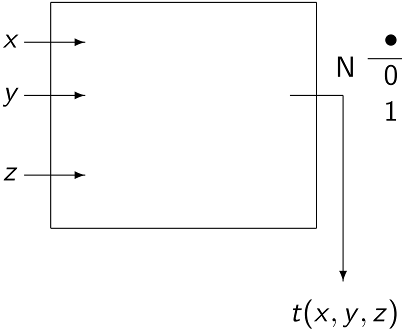
N

•	0	1
0	1	1
1	1	0

~~Performance Table $\Leftarrow$ Switching Circuit~~

Performance Specification

x	y	z	$t(x,y,z)$
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0



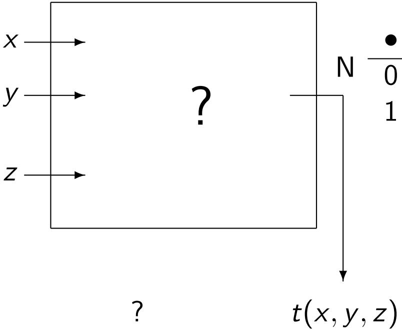
N

$\bullet$	0	1
0	1	1
1	1	0

~~Performance Table $\Leftarrow$ Switching Circuit~~

Performance Specification

x	y	z	$t(x,y,z)$
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0



N

•	0	1
0	1	1
1	1	0

~~Performance Table $\Leftarrow$ Switching Circuit~~

Performance Specification $\underset{?}{\Rightarrow}$ Switching Circuit

Sheffer's Theorem (1913). *Every* $0, 1$*-switching circuit performance specification can be realized by a circuit built only from nand gates.*

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Definition. An n-valued gate G is a **primal gate** if every switching circuit performance specification with values $\{0, 1, \ldots, n-1\}$ can be realized by a circuit built only from G-gates.

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Rosenberg-Rouseau Theorem (1967). *A gate* G *is primal if and only if it has no non-trivial subgates, automorphisms or congruences.*

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Davies' Theorem (1968). *The asymptotic probability that a gate is primal is* $\frac{1}{e} \approx 0.368$.

Performance Specification

x	y	z	$t(x,y,z)$
0	0	0	3
0	0	1	1
0	0	2	0
0	0	3	2
0	0	4	3
0	1	0	4
0	1	1	1
0	1	2	3
0	1	3	0
0	1	4	3
0	2	0	4
0	2	1	1
0	2	2	3
0	2	3	0
	. . .		. . .

Performance Specification

x	y	z	$t(x,y,z)$
0	0	0	3
0	0	1	1
0	0	2	0
0	0	3	2
0	0	4	3
0	1	0	4
0	1	1	1
0	1	2	3
0	1	3	0
0	1	4	3
0	2	0	4
0	2	1	1
0	2	2	3
0	2	3	0
	...		...

Question 1. *Given a primal gate, e.g.* $G_1 = \langle\{0,1,2,3,4\}, *\rangle$ *and an arbitrary specification, how can we design a switching circuit that will meet that specification?*

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0	0	3	2
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0	1	0	4
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0	1	2	3
0	1	3	0
0	1	4	3
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	...		...

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- A table has $5^3 = 125$ rows, so there are $5^{125} \approx 10^{87}$ performance specifications.

Performance Specification

x	y	z	$t(x,y,z)$
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0	0	1	1
0	0	2	0
0	0	3	2
0	0	4	3
0	1	0	4
0	1	1	1
0	1	2	3
0	1	3	0
0	1	4	3
0	2	0	4
0	2	1	1
0	2	2	3
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	. . .		. . .

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0	1	2	3
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Question 2. *For which primal gates is Evolutionary Computation successful?*

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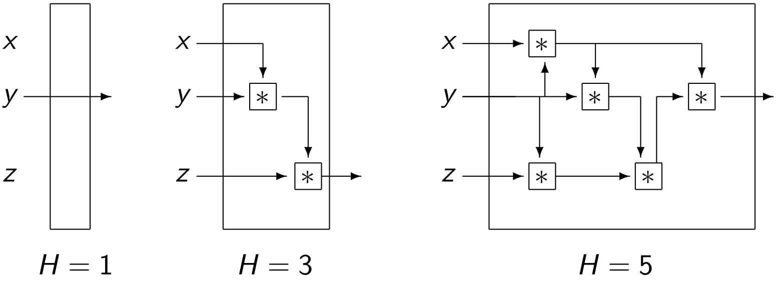
Question 2. *For which primal gates is Evolutionary Computation successful?*

Theorem. *A primal gate efficiently yields switching circuits from performance specifications by EC if it is asymptotically complete* (AC) *and has no separating relations* (NSR)*.*

The **height** H of a circuit is one more than the number of gates in the longest path from the output to an input.



Let $G = \langle\{0,1,2,3,4\},*\rangle$ be a binary gate and consider a fixed choice of inputs, say $\vec{d} = (3,1,2,0,4,4,2,2,2,2)$.

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$$\beta_0(H) := \mathrm{Prob}\langle t(\vec{d}) = 0 \mid t \text{ is a circuit of height at most } H \rangle$$

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For each input/output value q, we take $\beta_q(H)$ to be the probability that a G-circuit of height at most H with input $\vec{d}$ will output q.

Primal gate G_2:

Typical $\vec{d}$ for 10 inputs:

$$\vec{d} = (2, 0, 0, 3, 0, 1, 2, 0, 0, 1)$$

*	0	1	2	3	4
0	1	1	0	2	0
1	4	2	0	3	3
2	1	0	3	2	4
3	0	4	3	4	2
4	2	0	1	4	0

Primal gate G_2:

Typical $\vec{d}$ for 10 inputs:

$\vec{d} = (2, 0, 0, 3, 0, 1, 2, 0, 0, 1)$

*	0	1	2	3	4
0	1	1	0	2	0
1	4	2	0	3	3
2	1	0	3	2	4
3	0	4	3	4	2
4	2	0	1	4	0

H	$\beta_0(H)$	$\beta_1(H)$	$\beta_2(H)$	$\beta_3(H)$	$\beta_4(H)$
1	....	....	....	....	....
2					

Primal gate G_2:

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$$\vec{d} = (2, 0, 0, 3, 0, 1, 2, 0, 0, 1)$$

$*$	0	1	2	3	4
0	1	1	0	2	0
1	4	2	0	3	3
2	1	0	3	2	4
3	0	4	3	4	2
4	2	0	1	4	0

H	$\beta_0(H)$	$\beta_1(H)$	$\beta_2(H)$	$\beta_3(H)$	$\beta_4(H)$
1	0.500000	0.200000	0.200000	0.100000	0.000000
2					

$*$	0	1	2	3	4
0	1	1	0	2	0
1	4	2	0	3	3
2	1	0	3	2	4
3	0	4	3	4	2
4	2	0	1	4	0

Primal gate G_2:

Typical $\vec{d}$ for 10 inputs:

$$\vec{d} = (2, 0, 0, 3, 0, 1, 2, 0, 0, 1)$$

H	$\beta_0(H)$	$\beta_1(H)$	$\beta_2(H)$	$\beta_3(H)$	$\beta_4(H)$
1	0.500000	0.200000	0.200000	0.100000	0.000000
2	0.254545	0.427273	0.100000	0.081818	0.136364
3					

Primal gate G_2:

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$*$	0	1	2	3	4
0	1	1	0	2	0
1	4	2	0	3	3
2	1	0	3	2	4
3	0	4	3	4	2
4	2	0	1	4	0

H	$\beta_0(H)$	$\beta_1(H)$	$\beta_2(H)$	$\beta_3(H)$	$\beta_4(H)$
1	0.500000	0.200000	0.200000	0.100000	0.000000
2	0.254545	0.427273	0.100000	0.081818	0.136364
3	0.243315	0.212634	0.251693	0.111396	0.180962
5					

Primal gate G_2:

Typical $\vec{d}$ for 10 inputs:

$$\vec{d} = (2, 0, 0, 3, 0, 1, 2, 0, 0, 1)$$

$*$	0	1	2	3	4
0	1	1	0	2	0
1	4	2	0	3	3
2	1	0	3	2	4
3	0	4	3	4	2
4	2	0	1	4	0

H	$\beta_0(H)$	$\beta_1(H)$	$\beta_2(H)$	$\beta_3(H)$	$\beta_4(H)$
1	0.500000	0.200000	0.200000	0.100000	0.000000
2	0.254545	0.427273	0.100000	0.081818	0.136364
3	0.243315	0.212634	0.251693	0.111396	0.180962
5	0.275406	0.239708	0.168908	0.118508	0.197470
10	0.292228	0.236356	0.173358	0.121365	0.176693
15	0.292575	0.236033	0.173475	0.121482	0.176435
20	0.292581	0.236024	0.173479	0.121485	0.176431
700	0.292581	0.236024	0.173479	0.121485	0.176431

Definition. *A primal gate* G *is* **asymptotically complete** *if, for each input* $\vec{d}$*, each of the sequences*

$$\beta_0, \beta_1, \beta_2, \ldots$$

converges to positive number.

Definition. *A primal gate* G *is* **asymptotically complete** *if, for each input* $\vec{d}$*, each of the sequences*

$$\beta_0, \beta_1, \beta_2, \ldots$$

converges to positive number.

Is the gate G_2 asymptotically complete?

*	0	1	2	3	4
0	1	1	0	2	0
1	4	2	0	3	3
2	1	0	3	2	4
3	0	4	3	4	2
4	2	0	1	4	0

Try G_2 with $\vec{d} = (4, 4, 4, 4, 4, 4, 4, 4, 4, 4)$:

Try G_2 with $\vec{d} = (4, 4, 4, 4, 4, 4, 4, 4, 4, 4)$:

H	$\beta_0(H)$	$\beta_1(H)$	$\beta_2(H)$	$\beta_3(H)$	$\beta_4(H)$
1	0.000000	0.000000	0.000000	0.000000	1.000000
2	0.909091	0.000000	0.000000	0.000000	0.090909
3	0.091585	0.825764	0.082576	0.000000	0.000075
5	0.247629	0.183697	0.071840	0.477235	0.019599
10	0.294218	0.235343	0.172498	0.125556	0.172385
15	0.292666	0.235994	0.173471	0.121533	0.176336
23	0.292581	0.236024	0.173479	0.121485	0.176431
700	0.292581	0.236024	0.173479	0.121485	0.176431

Try G_2 with $\vec{d} = (4, 4, 4, 4, 4, 4, 4, 4, 4, 4)$:

H	$\beta_0(H)$	$\beta_1(H)$	$\beta_2(H)$	$\beta_3(H)$	$\beta_4(H)$
1	0.000000	0.000000	0.000000	0.000000	1.000000
2	0.909091	0.000000	0.000000	0.000000	0.090909
3	0.091585	0.825764	0.082576	0.000000	0.000075
5	0.247629	0.183697	0.071840	0.477235	0.019599
10	0.294218	0.235343	0.172498	0.125556	0.172385
15	0.292666	0.235994	0.173471	0.121533	0.176336
23	0.292581	0.236024	0.173479	0.121485	0.176431
700	0.292581	0.236024	0.173479	0.121485	0.176431

For G_2 with $\vec{d} = (2, 0, 0, 3, 0, 1, 2, 0, 0, 1)$, we obtained

700	0.292581	0.236024	0.173479	0.121485	0.176431

Conjecture 1. *If* G *is a primal gate and all* β *sequences converge to positive numbers for some input* $\vec{d}$*, then all* β *sequences converge to the same positive numbers for every input* $\vec{d}$*.*

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Conjecture 2. *The primal gate* G_2 *is asymptotically complete.*

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If this is conjecture is true, then we only need to check a single choice of $\vec{d}$ to decide if a primal gate is AC!

Conjecture 2. *The primal gate* G_2 *is asymptotically complete.*

Question. *Is every primal gate AC?*

$\bullet$	0	1
0	1	1
1	1	0

NAND N with $\vec{d} = (0, 1, 1, 0, 0, 1)$.

$\bullet$	0	1
0	1	1
1	1	0

NAND N with $\vec{d} = (0, 1, 1, 0, 0, 1)$.

H	$\beta_0(H)$	$\beta_1(H)$
1	0.50000	0.50000
2	0.31250	0.68750
3		
4		
5		
6		
7		

$\bullet$	0	1
0	1	1
1	1	0

NAND N with $\vec{d} = (0, 1, 1, 0, 0, 1)$.

H	$\beta_0(H)$	$\beta_1(H)$
1	0.50000	0.50000
2	0.31250	0.68750
3	0.47321	0.52679
4	0.27753	0.72247
5	0.52196	0.47804
6	0.22853	0.77148
7	0.59517	0.40483

NAND N with $\vec{d} = (0, 1, 1, 0, 0, 1)$.

$\bullet$	0	1
0	1	1
1	1	0

H	$\beta_0(H)$	$\beta_1(H)$	H	$\beta_0(H)$	$\beta_1(H)$	H	$\beta_0(H)$	$\beta_1(H)$
1	0.50000	0.50000	8	0.16388	0.83612	15	0.99308	0.00693
2	0.31250	0.68750	9	0.69909	0.30091	16	0.00005	0.99995
3	0.47321	0.52679	10	0.09055	0.90945	17	0.99990	0.00010
4	0.27753	0.72247	11	0.82710	0.17290	18	0.00000	1.00000
5	0.52196	0.47804	12	0.02989	0.97011	19	1.00000	0.00000
6	0.22853	0.77148	13	0.94111	0.05889	20	0.00000	1.00000
7	0.59517	0.40483	14	0.00347	0.99653	21	1.00000	0.00000

NAND N with $\vec{d} = (0, 1, 1, 0, 0, 1)$.

$\bullet$	0	1
0	1	1
1	1	0

H	$\beta_0(H)$	$\beta_1(H)$	H	$\beta_0(H)$	$\beta_1(H)$	H	$\beta_0(H)$	$\beta_1(H)$
1	0.50000	0.50000	8	0.16388	0.83612	15	0.99308	0.00693
2	0.31250	0.68750	9	0.69909	0.30091	16	0.00005	0.99995
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4	0.27753	0.72247	11	0.82710	0.17290	18	0.00000	1.00000
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7	0.59517	0.40483	14	0.00347	0.99653	21	1.00000	0.00000

This pattern was observed to continue to $H = 700$.

$\bullet$	0	1
0	1	1
1	1	0

NAND N with $\vec{d} = (0, 1, 1, 0, 0, 1)$.

H	$\beta_0(H)$	$\beta_1(H)$	H	$\beta_0(H)$	$\beta_1(H)$	H	$\beta_0(H)$	$\beta_1(H)$
1	0.50000	0.50000	8	0.16388	0.83612	15	0.99308	0.00693
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7	0.59517	0.40483	14	0.00347	0.99653	21	1.00000	0.00000

This pattern was observed to continue to $H = 700$.

Conjecture 3. *NAND is a primal gate that is NOT AC!*

A **binary relation** on a set $G = \{0, 1, 2, \ldots\}$ is a set $\sim$ of ordered pairs (a, b) of elements of G. Writing $a \sim b$ means (a, b) is in $\sim$.

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Example. The order relation $<$ on the set $G = \{0, 1, 2, 3, 4, 5\}$ is the binary relation

$$<:= \{(0,1),(0,2),(0,3),(0,4),(0,5),(1,2),(1,3),(1,4),\\ (1,5),(2,3),(2,4),(2,5),(3,4),3,5),(4,5)\}$$

A **binary relation** on a set $\mathrm{G} = \{0, 1, 2, \ldots\}$ is a set $\sim$ of ordered pairs (a, b) of elements of G. Writing $a \sim b$ means (a, b) is in $\sim$.

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$$(1,5),(2,3),(2,4),(2,5),(3,4),3,5),(4,5)\}$$

Relation Chart

for $<$

	b 0	1	2	3	4	5
0						
1						
a 2					•	
3						
4						
5						

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Relation Chart

for $<$

b

		0	1	2	3	4	5
	0		●	●	●	●	●
	1			●	●	●	●
a	2				●	●	●
	3					●	●
	4						●
	5						

Let $G = \{0, 1, 2, \ldots, n-1\}$ be a gate. A binary relation $\sim$ is a **separating relation** on G if, for all $a, b, c \in G$,

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$$\text{(iv)}\ a \sim b \ \Rightarrow \ a * c \sim b * c \ \text{ and } \ c * a \sim c * b.$$

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$\mathrm{G}_4 = \{0, 1, 2, 3, 4\}$

$*$	0	1	2	3	4
0	3	1	4	0	2
1	0	4	3	2	1
2	4	2	0	1	3
3	1	3	2	4	0
4	2	0	1	2	4

Separating Relation

	0	1	2	3	4
0					
1					
2					
3					
4					

Let $\mathrm{G} = \{0, 1, 2, \ldots, n-1\}$ be a gate. A binary relation $\sim$ is a **separating relation** on G if, for all $a, b, c \in \mathrm{G}$,

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0	3	1	4	0	2
1	0	4	3	2	1
2	4	2	0	1	3
3	1	3	2	4	0
4	2	0	1	2	4

Separating Relation

	0	1	2	3	4
0			•		
1					
2					
3					
4					

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$*$	0	1	2	3	4
0	3	1	4	0	2
1	0	4	3	2	1
2	4	2	0	1	3
3	1	3	2	4	0
4	2	0	1	2	4

Separating Relation

	0	1	2	3	4
0	×		•		
1		×			
2			×		
3				×	
4					×

Let $G = \{0, 1, 2, \ldots, n-1\}$ be a gate. A binary relation $\sim$ is a **separating relation** on G if, for all $a, b, c \in G$,

$$\text{(i)} \sim \neq \emptyset, \quad \text{(ii)}\ a \nsim a, \quad \text{(iii)}\ a \sim b \ \Rightarrow\ b \sim a$$
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1	0	4	3	2	1
2	4	2	0	1	3
3	1	3	2	4	0
4	2	0	1	2	4

Separating Relation

	0	1	2	3	4
0	×		•		
1		×			
2	•		×		
3				×	
4					×

Let $\mathrm{G} = \{0, 1, 2, \ldots, n-1\}$ be a gate. A binary relation $\sim$ is a **separating relation** on G if, for all $a, b, c \in \mathrm{G}$,

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*	0	1	2	3	4
0	3	1	4	0	2
1	0	4	3	2	1
2	4	2	0	1	3
3	1	3	2	4	0
4	2	0	1	2	4

$a = 1, \ b = 4, \ c = 3$

Separating Relation

	0	1	2	3	4
0	×		•		
1		×			
2	•		×		
3				×	
4					×

Let $\mathsf{G} = \{0, 1, 2, \ldots, n-1\}$ be a gate. A binary relation $\sim$ is a **separating relation** on G if, for all $a, b, c \in \mathsf{G}$,

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$*$	0	1	2	3	4
0	3	1	4	0	2
1	0	4	3	2	1
2	4	2	0	1	3
3	1	3	2	4	0
4	2	0	1	2	4

$a = 1, \ b = 4, \ c = 3$

Separating Relation

	0	1	2	3	4
0	×		•		
1		×			×
2	•		×		
3				×	
4					×

Let $\mathrm{G} = \{0, 1, 2, \ldots, n-1\}$ be a gate. A binary relation $\sim$ is a **separating relation** on G if, for all $a, b, c \in \mathrm{G}$,

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0	3	1	4	0	2
1	0	4	3	2	1
2	4	2	0	1	3
3	1	3	2	4	0
4	2	0	1	2	4

$a = 1,\ b = 4,\ c = 3$

Separating Relation

	0	1	2	3	4
0	×		•		
1		×			×
2	•		×		
3				×	
4		×			×

Let $\mathrm{G} = \{0, 1, 2, \ldots, n-1\}$ be a gate. A binary relation $\sim$ is a **separating relation** on G if, for all $a, b, c \in \mathrm{G}$,

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$\mathrm{G}_5 = \{0, 1, 2, 3, 4\}$

*	0	1	2	3	4
0	3	1	4	0	2
1	0	4	3	2	1
2	4	2	0	1	3
3	1	3	2	4	0
4	2	0	1	3	4

Let $\mathrm{G} = \{0, 1, 2, \ldots, n-1\}$ be a gate. A binary relation $\sim$ is a **separating relation** on G if, for all $a, b, c \in \mathrm{G}$,

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*	0	1	2	3	4
0	3	1	4	0	2
1	0	4	3	2	1
2	4	2	0	1	3
3	1	3	2	4	0
4	2	0	1	3	4

Latin Square

Let $\mathrm{G} = \{0, 1, 2, \ldots, n-1\}$ be a gate. A binary relation $\sim$ is a **separating relation** on G if, for all $a, b, c \in \mathrm{G}$,

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$\mathrm{G}_5 = \{0, 1, 2, 3, 4\}$

*	0	1	2	3	4
0	3	1	4	0	2
1	0	4	3	2	1
2	4	2	0	1	3
3	1	3	2	4	0
4	2	0	1	3	4

Latin Square

$\neq$ is a Separating Relation

	0	1	2	3	4
0		•	•	•	•
1	•		•	•	•
2	•	•		•	•
3	•	•	•		•
4	•	•	•	•	

*	0	1	2	3	4
0					
1					
2					
3					
4					

Bill's Gate

*	0	1	2	3	4
0	2	0	2	0	3
1	1	3	4	3	2
2	2	4	4	3	2
3	2	2	4	0	1
4	1	2	1	2	4

Bill's Gate

*	0	1	2	3	4
0	2	0	2	0	3
1	1	3	4	3	2
2	2	4	4	3	2
3	2	2	4	0	1
4	1	2	1	2	4

	0	1	2	3	4
0					
1					
2					
3					
4					

Bill's Gate					
*	0	1	2	3	4
0	2	0	2	0	3
1	1	3	4	3	2
2	2	4	4	3	2
3	2	2	4	0	1
4	1	2	1	2	4

In NO Separating Relation					
	0	1	2	3	4
0					
1					
2					
3					
4					

Bill's Gate

*	0	1	2	3	4
0	2	0	2	0	3
1	1	3	4	3	2
2	2	4	4	3	2
3	2	2	4	0	1
4	1	2	1	2	4

In NO Separating Relation

	0	1	2	3	4
0					
1					
2					
3					
4					

Theorem. *Bill's Gate has No Separating Relations.*

G_6

*	0	1	2	3	4
0	3	1	4	0	2
1	0	3	3	2	1
2	4	2	0	1	3
3	1	3	2	4	0
4	2	0	1	3	4

G_6

*	0	1	2	3	4
0	3	1	4	0	2
1	0	3	3	2	1
2	4	2	0	1	3
3	1	3	2	4	0
4	2	0	1	3	4

In NO Separating Relation

	0	1	2	3	4
0					
1					
2					
3					
4					

G_6

*	0	1	2	3	4
0	3	1	4	0	2
1	0	3	3	2	1
2	4	2	0	1	3
3	1	3	2	4	0
4	2	0	1	3	4

In NO Separating Relation

	0	1	2	3	4
0					
1					
2					
3					
4					

Theorem. *The gate* G_6 *has No Separating Relations.*

Statistical Test.

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Conjecture 4. *Almost all primals are Asymptotically Complete.*

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Conjecture 4. *Almost all primals are Asymptotically Complete.*

Conjecture 5. *Almost all primals have No Separating Relations.*

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Conjecture 4. *Almost all primals are Asymptotically Complete.*

Conjecture 5. *Almost all primals have No Separating Relations.*

If these conjectures are true, then evolutionary methods will find circuit designs for almost all primal gates.

Thanks for listening!

— David

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