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EVOLUTION OF ALGEBRAIC TERMS 1: TERM TO TERM OPERATION CONTINUITY

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This study was inspired by recent successful applications of evolutionary computation to the problem of finding terms to represent arbitrarily given operations on a primal groupoid. Evolution requires that small changes in a term result in small changes in the associated term operation. We prove a theorem giving two readily testable conditions under which a groupoid must have this continuity property, and offer evidence that most primal groupoids satisfy these conditions.

Keywords: Evolutionary computation; term generation; term operation; primal algebras.

Mathematics Subject Classification 2010: 08A40, 68Q05, 92D15

0. Introduction

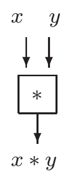
The 1913 seminal paper of Sheffer [9] introduced the NAND operation $*$ on $\{0,1\}$ that has ever since been used as a basis for electronic switching circuits. His groupoid $\mathbf{S} := \langle\{0,1\}, *\rangle$ in Fig. 1 is implemented below it as a physical NAND gate, taking $0,1$-valued inputs x and y and outputting $x * y$. By interconnecting NAND gates, it is possible to build a complex circuit that takes an arbitrary sequence of $0,1$-valued inputs, for example (x, y, z), and gives either 0 or 1 as output.

The significant property of $\mathbf{S}$ is that, for every $k \in \mathbb{N}$ and every k-ary performance specification $g\colon \{0,1\}^k \to \{0,1\}$, that is, every k-ary operation on $\{0,1\}$, there is a switching circuit that will compute the values of g. For example, in Fig. 1 the values of the ternary operation g are calculated by the switching circuit on the right. The *groupoid term* $t(x, y, z)$ codes a design for this switching circuit.

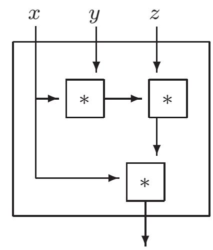
In practice, it turned out to be important to build switching circuits that calculate with more than two input–output values. This meant finding finite groupoids $\mathbf{G} = \langle\{0, 1, \ldots, n-1\}, *\rangle$ with the property that every operation on its underlying

S

$*$	0	1
0	1	1
1	1	0



x	y	z	$g(x,y,z)$
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	0



$t(x,y,z) = x * ((x * y) * z)$

Fig. 1. A switching circuit built from NAND gates.

set is realized by some groupoid term, and therefore some switching circuit. Groupoids with this property are called *primal*. Over time many were found, culminating in the theorem of Rosenberg [7], Quackenbush [6] and its corollary of Rouseau [8] which demonstrate that primal groupoids are common and tell us how to identify them.

An interesting twist to these and other known results is that they show that certain groupoids are primal without offering any practical guidelines as to how to find the groupoid terms coding the switching circuits that they guarantee to exist. As a centennial observance of Sheffer's work, we offer here a new and promising approach to this seemingly intractable problem. It is based on the relatively new and powerful technique of evolutionary computation.

Evolutionary computation is a general strategy of finding solutions to problems by an electronic simulation of biological evolution (see [3]). It depends on having some efficiently computable measure of the "fitness" of a candidate solution. A typical system begins with a large population of randomly constructed candidate solutions, and then iterates the following cycle: measure the fitness of each candidate, select a portion of these candidates in a way that is biased toward higher fitness, fill out a new population with mutations and recombinations of the selected candidates, and repeat. When successful, this process will evolve the population toward a progressively higher maximum fitness. It terminates if a fully fit solution is found.

Successful application of evolutionary computation relies heavily on the choice of a suitable problem. A candidate solution to the problem typically has two forms. It has a symbolic design, the *genotype*, describing the solution itself, e.g. a particular DNA sequence. It also has a realization, the *phenotype*, which is the outcome of that solution, e.g. the resulting organism. A mutation is a change in the genotype which then may result in a change in the fitness of the phenotype. Success requires that we have suitable metrics on the space of genotypes and the space of phenotypes that are continuously connected. This requirement can be roughly stated as a condition on a problem that is necessary for it to admit an evolutionary solution.

Continuity Condition. *The map taking genotypes to phenotypes is continuous in the sense that small mutations in genotypes normally lead to small changes in the fitness of their phenotypes.*

For example, DNA coding needs to be sufficiently robust that small changes in the coded sequence can be made without reducing fitness to the point that the organism cannot survive.

Each year since 2004 the Association for Computing Machinery has held its annual Genetic and Evolutionary Computation Conference (GECCO) featuring evolutionary computation discoveries that meet a specific definition of being "human competitive". The intent is that these discoveries would have been viewed as impressive achievements had they been made by humans. The 2008 entry [10] of Spector, Clark, Barr, Klein and Lindsay used evolutionary methods to find terms for arbitrarily given operations on 3- and 4-element groupoids known by Rouseau's Theorem [8] to be primal. In Fig. 2, we illustrate two groupoids presented in [10], together with the two target operations they sought to find by evolutionary computation.

In [10], the authors justified the "human competitive" designation of this problem by looking at the two best available alternative methods using the ternary discriminator operation and a majority operation as common yardsticks to compare results. One alternative is to ask Freese's *UACalc* program [4] to do an exhaustive search, starting with the smallest terms and working up. This method is guaranteed to always find a minimal length term. The other is the Primality Test of Clark, Davey, Pitkethly and Rifqui [1], which is the most computationally efficient known algebraic method to give small building block terms from which a term for an arbitrarily given operation can be recursively constructed. For groupoids in this size range, *UACalc* instantly delivers these building block terms.

The discriminator operation of $\mathbf{A}_1$ must be found in the search space of all $3^{3^3} \approx 10^{13}$ ternary operations on $\{0,1,2\}$. Although *UACalc* will give a minimal discriminator term, the authors of [10] calculated that a continuous computation would take an expected time to success of about one month. Although *UACalc* gives Primality Test terms instantly, they calculated that the required recursions would build a term with over 10^7 variable occurrences. In contrast, evolutionary

$\mathbf{A}_1$

$*$	0	1	2
0	2	1	2
1	1	0	0
2	0	0	1

$\mathbf{B}_1$

$*$	0	1	2	3
0	1	3	1	0
1	3	2	0	1
2	0	1	3	1
3	1	0	2	0

$$d(a,b,c) := \begin{cases} c & \text{if } a = b \\ a & \text{if } a \neq b \end{cases}$$

Discriminator operation

$$m(a,b,b) = m(b,a,b) = m(b,b,a) = b$$

Majority operation

Fig. 2. Primal algebras and their target operations.

computation produced the following 40-variable term in under five minutes:

$$
\begin{aligned}
t&(x,y,z)\\
&= (((((((((x*(y*x))*x)*z)*(z*x))*((x*(z*(x*(z*y))))*z))*z)\\
&\quad *z)*(z*((((x*(((z*z)*x)*(z*x)))*x)*y)*(((y*(z*(z*y)))\\
&\quad *(((y*y)*x)*z))*(x*(((z*z)*x)*(z*(x*(z*y))))))))).
\end{aligned}
$$

A majority term for $\mathbf{B}_1$ comes from a vastly larger search space. There are 40 majority triples $(a,b,b),(b,a,b),(b,b,a)$ and therefore a search space of $4^{40} \approx 10^{24}$ maps from the majority triples to B_1. In [10], it is shown that a continuous exhaustive search for a majority term would have an expected time to success of about 10^{10} *years*. The recursions used by the Primality Test construction would build a term with over 10^{24} variables. (See also [2].) Again evolutionary computation produced a human scale term in human scale time. For the majority operation on $\mathbf{B}_1$ evolutionary computation produced a 64-variable term in under two hours:

$$
\begin{aligned}
m&(x,y,z)\\
&= ((y*(((x*x)*((y*(((x*x)*z)*y))*(x*(((x*x)*((z*(((((z*y)\\
&\quad *y)*z)*((z*y)*(z*y)))*y))*((x*x)*(((x*x)*y)*y))))*x))))\\
&\quad *y))*(x*(((x*x)*((y*(((x*x)*y)*y))*(x*(x*((y*(((x*x)*z)\\
&\quad *y))*(x*(((x*x)*(((y*y)*((((z*(z*x))*y)*(x*(z*y)))*((y*x)\\
&\quad *x)))*(x*(x*x))))*x)))))))*x))).
\end{aligned}
$$

The authors of [10] were pleased to accept the first place in the 2008 GECCO competition for these results.

This study was motivated by the conviction that these successes would not have been possible if these groupoids did not satisfy the continuity condition, with terms as genotypes and term operations as phenotypes. This raises the question as to which finite groupoids do satisfy this condition. Is satisfaction of the continuity condition frequent or rare? Suppose we take for the groupoid a finite group and consider a very large groupoid term; then change just one variable occurrence to an occurrence of a different variable. Continuity would predict that the term operation is unlikely to change, whereas we know that, in fact, the term operation is guaranteed to change on every input sequence where the two variables differ. Does this not tell us that no non-trivial group is term continuous? Similarly, suppose we use a left-zero semigroup ($a*b := a$) and change only one variable occurrence. If it is the left most variable, the term operation will change dramatically. If not, it will not change at all. Does this violate continuity? To answer these questions we need a precise definition of "term continuity" in this context.

This paper introduces three concepts: *term continuity*, *separating relations* and *asymptotic completeness*. Term continuity, the continuity condition tailored to the term generation problem, will be defined in Sec. 1 where we give three equivalent formulations of this notion. In Sec. 2, we introduce separating relations on a finite groupoid and give an efficient algorithm to test for their presence. Asymptotic completeness is introduced in Sec. 3 together with a recursive tool to test a finite groupoid for this property. These ideas come together in Sec. 4, where we prove the Continuity Theorem 24 stating that a finite asymptotically complete groupoid is term continuous if and only if its subgroupoids have no separating relations. Many examples suggest that the conditions of this theorem are met by most finite groupoids, making term continuity much more the rule than the exception.

In the upcoming paper [2], we will give experimental evidence that the algorithm of [10] will succeed if and only if the groupoid satisfies the continuity condition. We will also present a new co-evolutionary algorithm that efficiently produces terms for arbitrarily given operations on groupoids with much larger search spaces. We will make a compelling case that this algorithm will succeed if and only if the groupoid satisfies the continuity condition and is idemprimal.

1. The Continuity Condition for Term Generation

In this section, we fix notation and terminology with the goal of presenting a precise mathematical formulation of the continuity condition in the context of terms on a finite groupoid.

Throughout this study, we let $\mathbb{N}$ be the set of positive integers and let $\mathbb{N}^{+1} := \{m+1 \mid m \in \mathbb{N}\}$ be the integers greater than 1. We denote the cardinality of a finite set X by $\|X\|$. If Y is the set of members of X having property P, we write

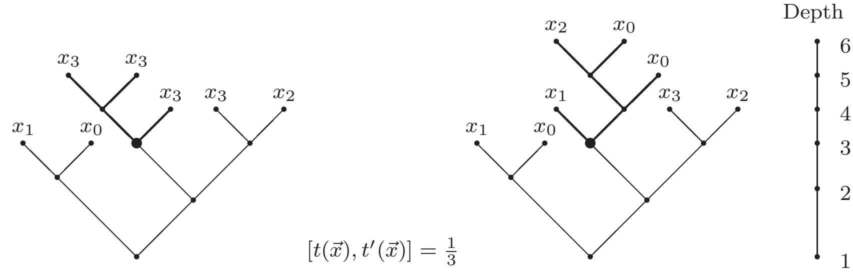
$$Y := \{x \in X \mid P(x)\}.$$

The probabilities we use will all be computed over some finite domain X carrying the uniform probability distribution. Thus, if P is a property, then

$$\mathrm{Prob}\,\langle P(x) : x \in X\rangle$$

denotes the proportion of members of X that have property P, also written as $\|\{x \in X \mid P(x)\}\|/\|X\|$. Throughout $\mathbf{G} := \langle G, *\rangle$ will be a finite groupoid with $G = \{0, 1, \ldots, n-1\}$.

We will study terms in a single binary operation symbol $*$. Hereafter, unless otherwise stated, "term" will mean a term in the variables $\vec{x} := (x_0, x_1, \ldots, x_{k-1})$ for some fixed $k \in \mathbb{N}^{+1}$. The *depth* of a subterm occurrence in a term refers to the depth of its node in the term tree, with the depth of the root being 1. (See Fig. 3.) The *height* of a term is the greatest depth of its variables. For $H \in \mathbb{N}$, we denote by $\mathcal{T}_H$ the set of all terms of height at most H, and by $\mathcal{T}$ the set of all terms. A term $t(\vec{x}) \in \mathcal{T}$ defines a term operation $t^{\mathbf{G}} : G^k \to G$ where, for $\vec{d} \in G^k$, the value of $t^{\mathbf{G}}(\vec{d})$ is obtained by substituting d_i for x_i in $t(\vec{x})$ and the operation of $\mathbf{G}$ for $*$. Often we will omit the superscript "$\mathbf{G}$" on $t^{\mathbf{G}}$.



$$t(\vec{x}) = (x_1x_0)(\underbrace{((\mathbf{x_3x_3})\mathbf{x_3})}_{u_{t,t'}(\vec{x})}(x_3x_2)) \qquad t'(\vec{x}) = (x_1x_0)(\underbrace{(\mathbf{x_1}((\mathbf{x_2x_0})\mathbf{x_0}))}_{v_{t,t'}(\vec{x})}(x_3x_2))$$

Fig. 3. Distance measure on the term space $\mathcal{T}$.

Given $k \in \mathbb{N}^{+1}$ we will denote by G^{G^k} the set of all k-ary operations $q : G^k \to G$ on G. For $r, s \in G^{G^k}$, the *Hamming distance* is given by

$$\mathrm{HD}(r, s) = \|\{\vec{d} \in G^k \mid r(\vec{d}) \neq s(\vec{d})\}\| \leq n^k.$$

Under Hamming distance, it is easy to see that G^{G^k} forms a metric space.

In order to discuss continuity, we need a corresponding means of measuring distance in the space $\mathcal{T}$ of k-ary terms. Let $t, t' \in \mathcal{T}$. If $t = t'$, we define the distance $[t, t'] = 0$. If $t \neq t'$, we define the distance $[t, t'] = 1/D$, where D is the depth of the unique smallest subterms u of t and v of t' such that t' can be obtained from t by replacing this occurrence of u with v. This distance measure, which is not a metric, is illustrated by the example in Fig. 3 and is fully justified by Lemma 1.

To substantiate the existence of the required terms u and v, we consider terms $t \neq t'$ and recursively define subterms $u_{t,t'}$ of t and $v_{t,t'}$ of t' as follows. For a term $w \in \mathcal{T}\backslash\mathcal{T}_1$, let $w_L, w_R \in \mathcal{T}$ with $w = w_L w_R$. Let H_0 be the minimum of the heights of t and t'. If $H_0 = 1$, we define $u_{t,t'} = t$ and $v_{t,t'} = t'$. Assume $H_0 > 1$. Then $t = t_L t_R$ and $t' = t'_L t'_R$. Since $t \neq t'$, either $t_L \neq t'_L$ and $t_R \neq t'_R$, or only $t_L \neq t'_L$, or only $t_R \neq t'_R$.

- If both $t_L \neq t'_L$ and $t_R \neq t'_R$, let $u_{t,t'} = t$ and $v_{t,t'} = t'$.
- If only $t_L \neq t'_L$, let $u_{t,t'} = u_{t_L,t'_L}$ and $v_{t,t'} = v_{t_L,t'_L}$.
- If only $t_R \neq t'_R$, let $u_{t,t'} = u_{t_R,t'_R}$ and $v_{t,t'} = v_{t_R,t'_R}$.

Then two facts follow by induction on the minimal height H_0.

Lemma 1. *If $t, t' \in \mathcal{T}$ and $t \neq t'$, then*

(i) *t' can be obtained from t by replacing $u_{t,t'}$ with $v_{t,t'}$, and*
(ii) *if u and v are subterms of t and t', respectively, and t' can be obtained from t by replacing u with v, then $u_{t,t'}$ is a subterm of u and $v_{t,t'}$ is a subterm of v.*

To describe mutations, we say that a set $\mathcal{V} \subseteq \mathcal{T}$ is a *mutation set* if it is finite and it contains the set $\mathcal{T}_1$ of variables. We now introduce a new variable $\Diamond$ (the *lozenge*) which we think of as representing a missing subterm. A $\Diamond$-*term* is a term in variables $x_0, x_1, \ldots, x_{k-1}, \Diamond$ in which $\Diamond$ occurs exactly once. For each $H \in \mathbb{N}$, let $\mathcal{U}_H$ be the set of all pairs $(t(\vec{x}, \Diamond), u(\vec{x}))$ such that $t(\vec{x}, \Diamond)$ is a $\Diamond$-term, $u(\vec{x})$ is a term, and $t(\vec{x}, u(\vec{x})) \in \mathcal{T}_H$. Thus the pairs of $\mathcal{U}_H$ correspond exactly to the nodes in term trees of terms in $\mathcal{T}_H$. Relative to a fixed choice $\mathcal{V}$ of mutation set, we define

$$\mathcal{M}_H := \mathcal{U}_H \times \mathcal{V} = \{(t, u, v) \mid (t, u) \in \mathcal{U}_H \text{ and } v \in \mathcal{V}\},$$

where we omit the extra parentheses in the abbreviated name "(t, u, v)" for $((t, u), v)$. A triple $M := (t, u, v) = (t(\vec{x}, \Diamond), u(\vec{x}), v(\vec{x}))$ is a *mutation* if it is in $\mathcal{M}_H$ for some $H \in \mathbb{N}$. We think of M as replacing the subterm $u(\vec{x})$ of $t(\vec{x}, u(\vec{x}))$ with $v(\vec{x})$ to form the term $t(\vec{x}, v(\vec{x}))$. Note that each $\mathcal{M}_H$ is finite because $\mathcal{V}$ is finite, allowing us to compute sums and probabilities over $\mathcal{M}_H$.

Given a mutation (t, u, v) we will often use the abbreviations $t(u) := t(\vec{x}, u(\vec{x}))$ and $t(v) := t(\vec{x}, v(\vec{x}))$. The *size* of the mutation $M = (t, u, v)$ is $1/D$ where D is the depth of $\Diamond$ in $t(\vec{x}, \Diamond)$. Consequently, a mutation is *small* if and only if it occurs *deep* in the mutated term. Notice that, by Lemma 1(ii), the distance $[t(u), t(v)]$ is at most the size of M. Given these notions we can now make a first pass at a formulation of the continuity condition for the problem of term generation.

`Continuity Condition` (draft formulation). *For $k \in \mathbb{N}^{+1}$, a finite groupoid $\mathbf{G}$ is* k-term continuous *if changes deep in a k-ary term result in correspondingly small changes in the term operation.*

There are still two shortcomings to this definition. One is that Hamming distance has only integer values from 0 to n^k, so it is not possible to make arbitrarily small changes in the term function. Also, it is not necessary that every small change in a term make a small change in the term operation. Evolution is a process involving large numbers of term interactions. As a result, we only need to know that this is true with sufficiently high probability. To resolve these shortcomings we define, relative to a fixed choice of k and $\mathbf{G}$, the *Hamming distance* of the mutation $M = (t, u, v)$ to be the Hamming distance between the term operations defined by $t(u)$ and $t(v)$ in the space G^{G^k}:

$$\mathrm{HD}(M) := \|\{\vec{d} \in G^k \mid t(\vec{d}, u(\vec{d})) \neq t(\vec{d}, v(\vec{d}))\}\|.$$

Both shortcomings are resolved by using the *mean* Hamming distance over $\mathcal{M}_H$,

$$\mu\mathrm{HD}(H) := \frac{1}{\|\mathcal{M}_H\|} \sum \{\mathrm{HD}(M) \mid M \in \mathcal{M}_H\}.$$

We can now give a precise formulation of the continuity condition for the term generation problem.

Continuity Condition. *For* $k \in \mathbb{N}^{+1}$, *a finite groupoid* $\mathbf{G}$ *is* k-term continuous *provided that*

$$\lim_{H\to\infty} \mu\mathrm{HD}(H) = 0$$

for every mutation set $\mathcal{V}$. *In words this says that, on average, changes deep in a* k*-ary term result in correspondingly small changes in the term operation.*

Our goal is to determine for which choices of the groupoid $\mathbf{G}$ and the arity $k \in \mathbb{N}^{+1}$ this condition holds. As an illustration of these ideas, consider the left-zero semigroup $\mathbf{L} := \langle\{0, 1, \ldots, n-1\}, *\rangle$ defined by $a * b := a$ for all $a, b \in L$. For this example every mutation of a term that changes the left most variable will change the final value for every choice of $\vec{d} \in L^k$ that assigns different values to the old and new left most variables. Because of this the draft formulation of the Continuity Condition is not satisfied. However, it seems likely that most mutations of terms in $\mathcal{T}_H$ will not affect the left most variable, and therefore will not affect the final value at all. If this is correct, we would expect that the mean $\mu\mathrm{HD}(H)$ would approach 0 so that the Continuity Condition would be fulfilled. Example 25 will use our main theorem to show that this is indeed correct.

Continuity is connected to evolution through the notion of fitness. In order for evolution to apply to terms, it must be that small changes in terms generally result in small changes in the *fitness* of their term operations. To make this requirement precise, we state the evolutionary goal in a somewhat more general setting that will be extensively exploited in [2]. By a k-ary *array* over G, we mean a map $A : G^k \to 2^G\backslash\{\varnothing\}$, that is, a G^k-indexed sequence of non-empty subsets of G. By a *solution* to an array A, we mean a term $t(\vec{x})$ such that, for all $\vec{d} \in G^k$, we have $t(\vec{d}) \in A(\vec{d})$. For example, if $g : G^k \to G$ is a k-ary operation on G, we denote by $[g]$ the array whose value at $\vec{d}$ is $\{g(\vec{d})\}$. Then a term represents the operation $g : G^k \to G$ if and only if it is a solution to the array $[g]$. As a different example, let A be the 3-ary array over $B_1 = \{0, 1, 2, 3\}$ of Fig. 2 defined by

$$A(a, b, c) := \begin{cases} \{a\} & \text{if } a = b \text{ or } a = c, \\ \{b\} & \text{if } b = c; \\ B_1 & \text{otherwise.} \end{cases}$$

Then $m(x_0, x_1, x_2)$ is a majority term for $\mathbf{B}_1$ if and only if $m(x_0, x_1, x_2)$ is a solution to the array A.

Assume that A is a k-ary array over G that we would like to solve. The *fitness* of a k-ary term $t(\vec{x})$ as an approximation to a solution to A is

$$F_A(t) := \|\{\vec{d} \in G^k \mid t^{\mathbf{G}}(\vec{d}) \in A(\vec{d})\}\|.$$

Thus t is a solution to A if and only if $F_A(t) = n^k$. As a different description of fitness, define a k-ary operation t^A as follows. For each $\vec{d} \in G^k$ choose $t^A(\vec{d}) \in A(\vec{d})$ subject to the one constraint that $t^A(\vec{d}) = t^{\mathbf{G}}(\vec{d})$ if $t^{\mathbf{G}}(\vec{d}) \in A(\vec{d})$. Then we have

$$F_A(t) = n^k - \mathrm{HD}(t^{\mathbf{G}}, t^A).$$

The result of invoking a mutation $M = (t, u, v)$ to replace $t(\vec{x}, u(\vec{x}))$ by $t(\vec{x}, v(\vec{x}))$ is a change in fitness of

$$\Delta F_A(M) := |F_A(t(u)) - F_A(t(v))|.$$

For $H \in \mathbb{N}$, the mean change in fitness over all mutations in $\mathcal{M}_H$ is then given as

$$\mu\Delta F_A(H) := \frac{1}{\|\mathcal{M}_H\|} \sum \{\Delta F_A(M) \,|\, M \in \mathcal{M}_H\}.$$

The following theorem shows that the Continuity Condition provides what evolution needs of fitness. The proof requires that the *phenotype space* G^{G^k} be a metric space, which — conveniently — it is.

Fitness Theorem 2. *Let* $A : G^k \to 2^G \backslash \{\varnothing\}$ *be a target array. If the finite groupoid* $\mathbf{G}$ *is* k*-term continuous*, *then*

$$\lim_{H \to \infty} \mu\Delta F_A(H) = 0.$$

In words this conclusion says that, *on average*, *changes deep in a term will result in correspondingly small changes in the fitness of the term.*

Proof. For terms $r, s \in \mathcal{T}$, let $\Delta F_A(r, s) := |F_A(s) - F_A(r)|$ denote the difference in fitness. We first claim that $\Delta F_A(r, s) \leq \mathrm{HD}(r^{\mathbf{G}}, s^{\mathbf{G}})$. To see this, assume that $F_A(r) \leq F_A(s)$. Then $\mathrm{HD}(r^{\mathbf{G}}, r^A) \geq \mathrm{HD}(s^{\mathbf{G}}, s^A)$, and we have

$$\begin{aligned}
\Delta F_A(r, s) &= F_A(s) - F_A(r) \\
&= (n^k - \mathrm{HD}(s^{\mathbf{G}}, s^A)) - (n^k - \mathrm{HD}(r^{\mathbf{G}}, r^A)) \\
&= \mathrm{HD}(r^{\mathbf{G}}, r^A) - \mathrm{HD}(s^{\mathbf{G}}, s^A) \\
&\leq \mathrm{HD}(r^{\mathbf{G}}, s^A) - \mathrm{HD}(s^{\mathbf{G}}, s^A) \quad \text{(definition of } r^A\text{)} \\
&\leq \mathrm{HD}(r^{\mathbf{G}}, s^{\mathbf{G}}) \quad \text{(the triangle inequality).}
\end{aligned}$$

Then, for every mutation $M = (t, u, v)$,

$$\Delta F_A(M) := |F_A(t(u)) - F_A(t(v))| = \Delta F_A(t(u), t(v)) \leq \mathrm{HD}(M).$$

Averaging over all mutations in $\mathcal{M}_H$, we have $\mu\Delta F_A(H) \leq \mu\mathrm{HD}(H)$. Since $\mathbf{G}$ is k-term continuous, $\mu\mathrm{HD}(H) \to 0$ as $H \to \infty$ and consequently $\mu\Delta F_A(H) \to 0$ as $H \to \infty$. □

We close this section with a theorem that gives three equivalent formulations for term continuity. The first is our *definition* of "term continuity". The second is a condition that can be verified for each $\vec{d} \in G^k$ individually and will be applied in Sec. 4 to *prove* that certain groupoids are term continuous. The third asserts that the likelihood that a subterm $u(\vec{x})$ of a term $t(\vec{x})$ will have any bearing whatsoever on the term operation $t(\vec{x})^{\mathbf{G}}$ diminishes to zero as the depth of the subterm $u(\vec{x})$ goes to infinity. This formulation will be applied in [2] to show how term continuity can be *used* to show that the search space size shrinks to one.

Theorem 3. *Let* $\mathbf{G}$ *be a finite groupoid, let* $k \in \mathbb{N}^{+1}$ *and let* $\mathcal{V}$ *be a mutation set. Then the following are equivalent. In particular,* $\mathbf{G}$ *is* k*-term continuous if and only if each one is true for all mutation sets* $\mathcal{V}$:

(i) $\lim_{H\to\infty} \mu\mathrm{HD}(H) = 0$.
(ii) $\lim_{H\to\infty} \mathrm{Prob}\,\langle t(\vec{d}, u(\vec{d})) \neq t(\vec{d}, v(\vec{d})) : (t,u,v) \in \mathcal{M}_H\rangle = 0$ *for every* $\vec{d} \in G^k$.
(iii) $\lim_{H\to\infty} \mathrm{Prob}\,\langle t(\vec{x}, u(\vec{x}))^{\mathbf{G}} = t(\vec{x}, v(\vec{x}))^{\mathbf{G}} : (t,u,v) \in \mathcal{M}_H\rangle = 1$.

Proof. We first prove the equivalence of (i) and (ii). The table in Fig. 4 lists the mutations in $\mathcal{M}_H$ down the side and the substitution sequences in G^k across the top. In the row of $M = (t,u,v)$ and column of $\vec{d}$ we put "$=$" if $t(\vec{d}, u(\vec{d})) = t(\vec{d}, v(\vec{d}))$ and "$\neq$" if not. The number of $\neq$'s in the row of $M \in \mathcal{M}_H$ is $\mathrm{HD}(M)$. The number of $\neq$'s in the column of $\vec{d} \in \mathbf{G}^k$ is $\|\mathcal{M}_H\| \cdot \mathrm{Prob}\,\langle t(\vec{d}, u(\vec{d})) \neq t(\vec{d}, v(\vec{d})) : (t,u,v) \in \mathcal{M}_H\rangle$. We prove the lemma by computing the total number of $\neq$'s by rows and then by columns.

$$\begin{aligned}
\mu\mathrm{HD}(H) &= \frac{1}{\|\mathcal{M}_H\|} \sum \{\mathrm{HD}(M) \mid M \in \mathcal{M}_H\} \\
&= \frac{1}{\|\mathcal{M}_H\|} \sum \{\|\{\vec{d} \in G^k \mid t(\vec{d}, u(\vec{d})) \neq t(\vec{d}, v(\vec{d}))\}\| \mid (t,u,v) \in \mathcal{M}_H\} \\
&= \frac{1}{\|\mathcal{M}_H\|} \sum \{\|\{(t,u,v) \in \mathcal{M}_H \mid t(\vec{d}, u(\vec{d})) \neq t(\vec{d}, v(\vec{d}))\}\| \mid \vec{d} \in G^k\} \\
&= \sum \{\mathrm{Prob}\,\langle t(\vec{d}, u(\vec{d})) \neq t(\vec{d}, v(\vec{d})) : (t,u,v) \in \mathcal{M}_H\rangle \mid \vec{d} \in G^k\}.
\end{aligned}$$

The conclusion follows, since this is a finite sum of non-negative values over G^k.

To prove the equivalence of (ii) and (iii), we observe that both are equivalent to

$$\lim_{H\to\infty} \mathrm{Prob}\,\langle t(\vec{d}, u(\vec{d})) = t(\vec{d}, v(\vec{d})) \quad \text{for all } \vec{d} \in G^k : (t,u,v) \in \mathcal{M}_H\rangle = 1$$

since G^k is finite. $\square$

The following observation is an immediate consequence of part (ii) of this theorem.

Corollary 4. *Every subgroupoid of a* k*-term continuous groupoid is* k*-term continuous.*

$$\begin{array}{cc|ccccc|}
 & & & & & G^k & \\
 & & \vec{d}_1 & \vec{d}_2 & \vec{d}_3 & \cdots & \vec{d}_{n^k} \\ \hline
 & M_1 & = & \neq & \neq & \cdots & = \\
 & M_2 & \neq & = & \neq & \cdots & \neq \\
\mathcal{M}_H & M_3 & \neq & = & = & \cdots & \neq \\
 & \vdots & \vdots & & & & \vdots \\ \hline
\end{array}$$

Fig. 4. Computation of $\mu\mathrm{HD}(H)$.

2. Separating Relations

We give here an easily tested necessary condition for k-term continuity. Corollary 7 was extended to Theorem 6 by David Hobby, who suggested the following definition. We say that $\sigma \subseteq G \times G$ is a *separating relation* if, for all $a, b, c \in G$,

(i) $\sigma \neq \varnothing$,
(ii) $(a, a) \notin \sigma$, and
(iii) if $(a, b) \in \sigma$, then $(ac, bc) \in \sigma$ and $(ca, cb) \in \sigma$.

For example, $\neq$ is a separating relation on any non-trivial quasigroup.

Lemma 5 (with D. Hobby). *If* $\mathbf{G}$ *has a separating relation, then* $\mathbf{G}$ *is not* k*-term continuous for any* $k \in \mathbb{N}^{+1}$.

Proof. Let σ be a separating relation on $\mathbf{G}$ and let $k \in \mathbb{N}^{+1}$. To use Theorem 3(ii) we must show that

$$\lim_{H\to\infty} \operatorname{Prob}\langle t(\vec{d}, u(\vec{d})) \neq t(\vec{d}, v(\vec{d})) : (t, u, v) \in \mathcal{M}_H\rangle \neq 0$$

for some $\vec{d} \in G^k$.

Since $\sigma \neq \varnothing$ and $k > 1$, we can choose $\vec{d} \in G^k$ so that $(d_0, d_1) \in \sigma$. Now consider $H \in \mathbb{N}$ and $(t, u, v) \in \mathcal{M}_H$. If $u = x_0$ and $v = x_1 \in \mathcal{V}$, then $(u(\vec{d}), v(\vec{d})) \in \sigma$. Since σ is a separating relation, $(t(\vec{d}, u(\vec{d})), t(\vec{d}, v(\vec{d}))) \in \sigma$ and therefore $t(\vec{d}, u(\vec{d})) \neq t(\vec{d}, v(\vec{d}))$. These implications can be expressed in terms of probabilities.

$$\begin{aligned}
&\operatorname{Prob}\langle t(\vec{d}, u(\vec{d})) \neq t(\vec{d}, v(\vec{d})) : (t, u, v) \in \mathcal{M}_H\rangle \\
&\qquad \geq \operatorname{Prob}\langle (t(\vec{d}, u(\vec{d})), t(\vec{d}, v(\vec{d}))) \in \sigma : (t, u, v) \in \mathcal{M}_H\rangle \\
&\qquad \geq \operatorname{Prob}\langle u = x_0 \text{ and } v = x_1 : (t, u, v) \in \mathcal{U}_H \times \mathcal{V}\rangle \\
&\qquad = \operatorname{Prob}\langle u = x_0 : (t, u) \in \mathcal{U}_H\rangle \cdot \operatorname{Prob}\langle v = x_1 : v \in \mathcal{V}\rangle \\
&\qquad = \operatorname{Prob}\langle u \in \mathcal{T}_1 : (t, u) \in \mathcal{U}_H\rangle \cdot \operatorname{Prob}\langle u = x_0 : u \in \mathcal{T}_1\rangle \cdot \frac{1}{\|\mathcal{V}\|} \\
&\qquad = \operatorname{Prob}\langle u \in \mathcal{T}_1 : (t, u) \in \mathcal{U}_H\rangle \cdot \frac{1}{k} \cdot \frac{1}{\|\mathcal{V}\|}.
\end{aligned}$$

To compute the probability that $u \in \mathcal{T}_1$, we first list the terms in $\mathcal{T}_H$. Under each $w \in \mathcal{T}_H$, we list all $(t, u) \in \mathcal{U}_H$ with $t(\vec{x}, u(\vec{x})) = w(\vec{x})$. If $w \in \mathcal{T}_H$, let s be the number of subterms of w and let v be the number of variable occurrences in w. By induction on H, we find that $2v = s + 1$. Thus $v/s = 1/2 + 1/2s > 1/2$. Since more than half of the subterms of each $w(\vec{x}) \in \mathcal{T}_H$ are in $\mathcal{T}_1$, we conclude

$$\operatorname{Prob}\langle t(\vec{d}, u(\vec{d})) \neq t(\vec{d}, v(\vec{d})) : (t, u, v) \in \mathcal{M}_H\rangle > \frac{1}{2} \cdot \frac{1}{k} \cdot \frac{1}{\|\mathcal{V}\|} = \frac{1}{2k\|\mathcal{V}\|},$$

and therefore the sequence is bounded away from zero. ☐

Theorem 6 (with D. Hobby). *If a homomorphic image of a subgroupoid of a finite groupoid* $\mathbf{G}$ *has a separating relation, then, for every* $k \in \mathbb{N}^{+1}$, *the groupoid* $\mathbf{G}$ *is not* k*-term continuous.*

Proof. Let $\mathbf{K}$ be a subgroupoid of $\mathbf{G}$, let $f : \mathbf{K} \to \mathbf{L}$ be a surjection, and let σ be a separating relation on $\mathbf{L}$. Then $\mathbf{L}$ is not k-term continuous, by Lemma 5. By Theorem 3(ii), since f is onto, there is a $\vec{d} \in K^k$ such that

$$\begin{aligned} 0 &\neq \lim_{H\to\infty} \operatorname{Prob}\langle t(f(\vec{d}), u(f(\vec{d}))) \neq t(f(\vec{d}), v(f(\vec{d}))) : (t,u,v) \in \mathcal{M}_H\rangle \\ &= \lim_{H\to\infty} \operatorname{Prob}\langle f(t(\vec{d}, u(\vec{d}))) \neq f(t(\vec{d}, v(\vec{d}))) : (t,u,v) \in \mathcal{M}_H\rangle \\ &\leq \lim_{H\to\infty} \operatorname{Prob}\langle t(\vec{d}, u(\vec{d})) \neq t(\vec{d}, v(\vec{d})) : (t,u,v) \in \mathcal{M}_H\rangle. \end{aligned}$$

Hence the final limit is not 0, and $\mathbf{G}$ is not k-term continuous by Theorem 3(ii). □

Corollary 7. *If a homomorphic image of a subgroupoid of a finite groupoid* $\mathbf{G}$ *is a non-trivial quasigroup, then* $\mathbf{G}$ *is not* k*-term continuous for any* $k \in \mathbb{N}^{+1}$.

Proof. This follows from Theorem 6 since $\neq$ is a separating relation on every non-trivial quasigroup. □

An exhaustive search of the 2^{n^2} binary relations on $\mathbf{G}$ shows that the presence of a separating relation is decidable. We give an efficient method to decide if separating relations are present that can easily be implemented by hand in small examples.

We say that $\rho \subseteq G^2$ is a *collapsing relation* if, for all $a, b, c \in G$,

(i) $(a, a) \in \rho$, and
(ii) if $(ac, bc) \in \rho$ or $(ca, cb) \in \rho$, then $(a, b) \in \rho$.

Since G^2 is a collapsing relation and the intersection of a non-empty collection of collapsing relations is again a collapsing relation, $\mathbf{G}$ has a unique minimal collapsing relation ρ_ω. We now give a construction for ρ_ω. Let

(1) $\rho_1 := \{(a, a) \mid a \in G\}$ and
(2) $\rho_{i+1} := \rho_i \cup \{(a, b) \in G^2 \mid \exists\, c \in G : (ca, cb) \in \rho_i \text{ or } (ac, bc) \in \rho_i\}$.

Let $\rho_\omega := \bigcup_{i=1}^{\infty} \rho_i$. The following facts are straightforward to verify.

Lemma 8. *Let* $\mathbf{G}$ *be a finite groupoid.*

(i) ρ_ω *is a collapsing relation, and is contained in every collapsing relation.*
(ii) *Each of* $\rho_1, \rho_2, \ldots$ *is symmetric and therefore* ρ_ω *is symmetric.*
(iii) *If* $(a, b) \in \rho_\omega$, *then* (a, b) *is not in any separating relation.*

Since $\mathbf{G}$ is finite, there is an $m \in \mathbb{N}$ such that

$$\rho_1 \subsetneq \rho_2 \subsetneq \rho_3 \subsetneq \cdots \subsetneq \rho_m = \rho_{m+1} = \rho_\omega.$$

These facts provide an efficient algorithm to decide if $\mathbf{G}$ has a separating relation.

Separating Relations Test 9. *Build the sequence* $\rho_1, \rho_2, \rho_3, \ldots$ *up to* ρ_ω.

(i) *If* $\rho_\omega = G^2$, *then* $\mathbf{G}$ *has no separating relations.*
(ii) *If* $\rho_\omega \neq G^2$, *then* $G^2 \backslash \rho_\omega$ *is a separating relation on* $\mathbf{G}$.

The following example illustrates both the ease of applying this algorithm and the ubiquity of groupoids with no separating relations. Experience has shown that this example is typical of most randomly generated groupoids, as might be expected since the value of π was not chosen by the author.

Example 10. Let $\mathbf{Pi} := \langle \{0,1,2,3,4\}, *\rangle$ be the groupoid whose table consists of the decimal digits of π reduced modulo 5. (See Fig. 5.) Then $\rho_3 = Pi^2$ and therefore $\mathbf{Pi}$ has no separating relations.

Proof. The collapsing table on the right side of Fig. 5 records the computation of ρ_ω, showing that $\rho_\omega = \rho_3 = Pi^2$. The table provides an entry for each ordered pair $(a,b) \in Pi^2$. Since each ρ_i is symmetric, we only record entries for $a \leq b$. We begin by entering ρ_1 on the diagonal, indicating that each pair (a,a) is in ρ_1. The remaining entries are coded as follows. If $(a,b) \notin \rho_i$ but $c \in Pi$ and $(ca, cb) \in \rho_i$, we enter "$\rho_{i+1} \,|\, c*$" in row a column b to indicate that c puts (a,b) into ρ_{i+1} from the left. Similarly, if $(a,b) \notin \rho_i$ but $c \in Pi$ and $(ac, bc) \in \rho_i$, we enter "$\rho_{i+1} \,|\, *c$" in row a column b to indicate that c puts (a,b) into ρ_{i+1} from the right. Having found that $\rho_3 = Pi^2$, we conclude that $\mathbf{Pi}$ has no separating relations. □

In Fig. 6, we give collapsing tables for $\mathbf{A}_1$ and $\mathbf{B}_1$, showing that they too have no separating relations. It appears that these examples are fairly typical, and that separating relations are rare. This is because many repetitions in the rows or columns make ρ_2 large and lead quickly to $\rho_\omega = G^2$. Some but few repetitions, like $ac = bc$ with $a \neq b$, also lead quickly to $\rho_\omega = G^2$. This is because few repetitions mean that

$*$	0	1	2	3	4
0	1	4	1	0	4
1	2	1	0	3	0
2	3	4	2	4	3
3	2	3	3	4	1
4	2	1	4	3	3

$\mathbf{Pi} := \langle \{0,1,2,3,4\}, *\rangle$

	0	1	2	3	4
0	ρ_1	$\rho_3\|0*$	$\rho_2\|0*$	$\rho_3\|1*$	$\rho_2\|2*$
1		ρ_1	$\rho_2\|3*$	$\rho_2\|2*$	$\rho_2\|0*$
2			ρ_1	$\rho_2\|*3$	$\rho_2\|1*$
3				ρ_1	$\rho_2\|4*$
4					ρ_1

Fig. 5. Collapsing table for $\mathbf{Pi}$: $\rho_\omega = \rho_3 = Pi^2$.

$\mathbf{A}_1$

	0	1	2
0	ρ_1	$\rho_2\|2*$	$\rho_2\|0*$
1		ρ_1	$\rho_2\|1*$
2			ρ_1

$\mathbf{B}_1$

	0	1	2	3
0	ρ_1	$\rho_3\|0*$	$\rho_2\|0*$	$\rho_2\|*0$
1		ρ_1	$\rho_2\|*3$	$\rho_2\|2*$
2			ρ_1	$\rho_3\|2*$
3				ρ_1

Fig. 6. Collapsing tables. $\mathbf{A}_1$: $\rho_\omega = \rho_2 = A_1^2$. $\mathbf{B}_1$: $\rho_\omega = \rho_3 = B_1^2$.

most rows and columns contain both a and b, making ρ_3 large. (See Example 28.) On the other hand, no repetitions means that we have a quasigroup and will never reach G^2.

3. Asymptotic Completeness

Our converse of Theorem 6 will require $\mathbf{G}$ to have a special asymptotic property in order to show that it is term continuous. Let $k \in \mathbb{N}^{+1}$ and $\vec{d} \in G^k$, and define $\mathbf{K}_{\vec{d}}$ to be the subgroupoid of $\mathbf{G}$ generated by the coordinates of $\vec{d}$. For $a \in G$, we define $\beta_{\vec{d},a} : \mathbb{N} \to [0,1]$ by

$$\beta_{\vec{d},a}(H) := \text{Prob}\,\langle t(\vec{d}) = a : t \in \mathcal{T}_H \rangle.$$

The *asymptotic residue* of $\vec{d}$ is the set $\underline{K}_{\vec{d}}$ of all $a \in G$ such that $\beta_{\vec{d},a}$ is eventually bounded away from 0, that is, there is a $\delta > 0$ and an $H_0 \in \mathbb{N}$ such that $\beta_{\vec{d},a}(H) > \delta$ for all $H > H_0$. Notice that $\underline{K}_{\vec{d}} \subseteq K_{\vec{d}}$ since, for all $a \in G \backslash K_{\vec{d}}$ and all $H \in \mathbb{N}$, we have $\text{Prob}\,\langle t(\vec{d}) = a : t \in \mathcal{T}_H \rangle = 0$. We say that $\mathbf{G}$ is *asymptotically k-complete* if $\underline{K}_{\vec{d}} = K_{\vec{d}}$ for all $\vec{d} \in G^k$. This is a completeness property because it says that every element of $\mathbf{G}$ that could possibly be in an asymptotic residue actually is.

Our analysis of the term sets $\mathcal{T}_H$ for $H \in \mathbb{N}$ will be aided by describing them recursively for each $k \in \mathbb{N}^{+1}$:

$$\mathcal{T}_1 \;=\; \{x_0, x_1, \ldots, x_{k-1}\} \quad \text{and}$$
$$\mathcal{T}_{H+1} := (\mathcal{T}_H * \mathcal{T}_H) \cup \mathcal{T}_1.$$

It follows that

$$\|\mathcal{T}_1\| = k \quad \text{and}$$
$$\|\mathcal{T}_{H+1}\| = \|\mathcal{T}_H\|^2 + k$$

for all $k \in \mathbb{N}^{+1}$. This recursive description of the term sets would smoothly lead to inductive proofs about terms were it not for the recurring presence of the variables $\mathcal{T}_1$. Our success will depend on the fact that $\mathcal{T}_1$ constitutes a rapidly diminishing proportion of $\mathcal{T}_H$ as H grows large. To quantify this fact, we name the two proportions of $\mathfrak{T}_{H+1}$, for each $H \in \mathbb{N}$, as

$$\iota_H := \frac{\|\mathfrak{T}_H * \mathfrak{T}_H\|}{\|\mathfrak{T}_{H+1}\|} = \frac{\|\mathfrak{T}_H\|^2}{\|\mathfrak{T}_H\|^2 + k} \quad \text{and} \quad o_H := \frac{k}{\|\mathfrak{T}_{H+1}\|} = \frac{k}{\|\mathfrak{T}_H\|^2 + k}.$$

Lemma 11. *For each $H \in \mathbb{N}$ and $k \in \mathbb{N}^{+1}$,*

(i) $$\iota_H + o_H = 1,$$

(ii) $$o_H < k^{-2^H+1},$$

(iii) $$\lim_{H\to\infty} \iota_H = 1,$$

(iv) $$\lim_{H\to\infty} o_H = 0.$$

Proof. Part (i) is immediate and (iii) and (iv) follow from (i) and (ii). To prove (ii) we show first that $\|\mathfrak{T}_{H+1}\| > k^{2^H}$. Clearly $\|\mathfrak{T}_{1+1}\| = k^2 + k > k^{2^1}$. If $\|\mathfrak{T}_{H+1}\| > k^{2^H}$, then $\|\mathfrak{T}_{H+2}\| > \|\mathfrak{T}_{H+1}\|^2 > (k^{2^H})^2 = k^{2^{H+1}}$. Then

$$o_H = \frac{k}{\|\mathfrak{T}_{H+1}\|} < \frac{k}{k^{2^H}} = k^{-2^H+1}. \qquad \square$$

Theorem 12. *If $k \in \mathbb{N}^{+1}$ and $\vec{d} \in G^k$, then $\underline{K}_{\vec{d}}$ is closed under $*$. Consequently, $\underline{K}_{\vec{d}} = \varnothing$ or $\underline{\mathbf{K}}_{\vec{d}}$ is a subgroupoid of $\mathbf{K}_{\vec{d}}$.*

Proof. Let $a, b \in \underline{K}_{\vec{d}}$ with $c := a * b$. Choose $H_0 \in \mathbb{N}$ and $\delta > 0$ such that $\beta_{\vec{d},a}(H)$ and $\beta_{\vec{d},b}(H)$ are greater than δ for all $H > H_0$. Using Lemma 11 we take $H_1 > H_0$ with $\iota_H > 1 - \delta$ if $H > H_1$. Now let $H > H_1$. For $t \in \mathfrak{T}_H * \mathfrak{T}_H$ let $t = t_L * t_R$. Then

$$\begin{aligned}
\beta_{\vec{d},c}(H+1) &= \mathrm{Prob}\,\langle t(\vec{d}) = c : t \in \mathfrak{T}_{H+1}\rangle \\
&\geq \mathrm{Prob}\,\langle t(\vec{d}) = c : t \in \mathfrak{T}_H * \mathfrak{T}_H\rangle \frac{\|\mathfrak{T}_H * \mathfrak{T}_H\|}{\|\mathfrak{T}_{H+1}\|} \\
&\geq \mathrm{Prob}\,\langle t_L(\vec{d}) = a \text{ and } t_R(\vec{d}) = b : t \in \mathfrak{T}_H * \mathfrak{T}_H\rangle \cdot \iota_H \\
&= \mathrm{Prob}\,\langle t_L(\vec{d}) = a : t \in \mathfrak{T}_H * \mathfrak{T}_H\rangle \cdot \mathrm{Prob}\,\langle t_R(\vec{d}) = b : t \in \mathfrak{T}_H * \mathfrak{T}_H\rangle \cdot \iota_H \\
&= \beta_{\vec{d},a}(H)\beta_{\vec{d},b}(H)\iota_H > \delta^2(1-\delta).
\end{aligned}$$

Consequently $c \in \underline{K}_{\vec{d}}$. $\square$

Corollary 13. *Let $\mathbf{G}$ be a finite groupoid with no proper subgroupoids and let $k \in \mathbb{N}^{+1}$. Then $\mathbf{G}$ is asymptotically k-complete if and only if the sequence $\beta_{\vec{d},a}$ is eventually bounded away from zero for all $\vec{d} \in G^k$ and all $a \in G$.*

Proof. If $\beta_{\vec{d},a}$ is eventually bounded away from zero for all $\vec{d} \in G^k$ and $a \in G$, then each $\underline{K}_{\vec{d}} = G = K_{\vec{d}}$ so $\mathbf{G}$ is asymptotically k-complete. Assume $\mathbf{G}$ is asymptotically k-complete and let $\vec{d} \in G^k$ and $a \in G$. As $\underline{K}_{\vec{d}} = K_{\vec{d}} \neq \varnothing$, it follows from Theorem 12 that $\underline{K}_{\vec{d}} = G$. Thus $a \in \underline{K}_{\vec{d}}$ and $\beta_{\vec{d},a}$ is eventually bounded away from zero. $\square$

We will see evidence that asymptotic completeness is very common. In fact, a finite groupoid with no proper subgroupoids that fails to be asymptotically complete must display a very peculiar oscillatory behavior instead.

Corollary 14. *Let* $\mathbf{G}$ *be a finite groupoid with no proper subgroupoid. Then* $\mathbf{G}$ *fails to be asymptotically* k*-complete if and only if there exist a* $\vec{d} \in G^k$*, an* $a \in G$*, a* $\delta > 0$ *and an increasing sequence* $J : \mathbb{N} \to \mathbb{N}$ *such that*

$$\lim_{m\to\infty} \beta_{\vec{d},a}(J_{2m}) = 0 \quad \textit{and} \quad \beta_{\vec{d},a}(J_{2m+1}) > \delta \quad \textit{for all } m \in \mathbb{N}.$$

Proof. Assume that $\mathbf{G}$ is not asymptotically k-complete. Then there is a $\vec{d} \in G^k$ such that $\underline{K}_{\vec{d}} \neq K_{\vec{d}}$. Since $\mathbf{G}$ has no proper subgroupoid, $\underline{K}_{\vec{d}} = \varnothing$ by Theorem 12. Since $\sum_{b\in G} \beta_{\vec{d},b}(H) = 1$ for each $H \in \mathbb{N}$, there must be an $a \in G$ such that $\beta_{\vec{d},a}$ does not converge to 0. Choose $\delta > 0$ such that, for all $H \in \mathbb{N}$, there is an $H' > H$ such that $\beta_{\vec{d},a}(H') > \delta$. Since $a \notin \underline{K}_{\vec{d}}$, it is also true that $\beta_{\vec{d},a}$ is not bounded away from 0. Using these two facts we can construct the required sequence J.

Now assume that there is a $k \in \mathbb{N}^{+1}$, a $\vec{d} \in G^k$, an $a \in G$ and a sequence J as described. Then $\beta_{\vec{d},a}$ neither converges to 0 nor is bounded away from 0. Since $\beta_{\vec{d},a}$ is not bounded away from 0, we have $a \notin \underline{K}_{\vec{d}}$. Since $a \in G = K_{\vec{d}}$, we conclude that $\mathbf{G}$ is not asymptotically k-complete. □

In contrast to the existence of separating relations, we do not know if asymptotic completeness is a decidable property of finite groupoids. Our proofs of asymptotic completeness are at this point limited to Examples 25 and 26 in Sec. 5. However, we will now give a recursive description of the $\beta_{\vec{d},a}$ sequences which can be used to efficiently compute them on a spreadsheet. This recursion will allow us to produce a test for asymptotic completeness that most finite groupoids appear to pass. It will require a few new tools.

We will need to consider probability distributions over $G = \{0, 1, \ldots, n-1\}$, that is, n-tuples $A = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{R}^n$ such that $\alpha_i \geq 0$ for all $i < n$ and $\alpha_0 + \alpha_1 + \cdots + \alpha_{n-1} = 1$. Let $\mathbb{P}_n$ denote the set of all probability distributions over G. We will denote points of $\mathbb{R}^n$ by capital letters and their coordinates by the corresponding lower case Greek letters. For $A, B \in \mathbb{R}^n$ we define the convolution operation $\otimes$ as $A \otimes B = C$ where

$$A = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1}), \quad B = (\beta_0, \beta_1, \ldots, \beta_{n-1}), \quad C = (\gamma_0, \gamma_1, \ldots, \gamma_{n-1}),$$

and, for $h < n$,

$$\gamma_h := \sum \{\alpha_i \beta_j \mid i * j = h \text{ with } i, j \in G\}.$$

A quick calculation shows that $\mathbb{P}_n$ is closed under $\otimes$. If $A, B \in \mathbb{P}_n$, then it is clear that the coordinates of $A \otimes B$ are non-negative. Summing the coordinates of $A \otimes B$, we have

$$\sum_{h\in G} \gamma_h = \sum_{h\in G} \sum_{i*j=h} \alpha_i \beta_j = \sum_{i,j\in G} \alpha_i \beta_j = \sum_{i\in G} \alpha_i \sum_{j\in G} \beta_j = 1 \cdot 1 = 1.$$

For $\vec{d} \in G^k$, each $H \in \mathbb{N} \cup \{0\}$ gives us the distribution

$$P_{\vec{d}}(H+1) := (\beta_{\vec{d},0}(H+1), \beta_{\vec{d},1}(H+1), \dots, \beta_{\vec{d},n-1}(H+1)) \in \mathbb{P}_n,$$

telling how likely it is that a term in $\mathfrak{T}_{H+1}$ with input $\vec{d}$ will return each value in G. Asymptotic completeness of $\mathbf{G}$ is a property of the sequences $\{P_{\vec{d}} \,|\, \vec{d} \in G^k\}$. It turns out that these sequences are closely approximated by convolutions. To see this, we denote by $\beta_{\vec{d},a}(H * H)$ the probability that $t(\vec{d}) = a$ where t ranges over the subset $\mathfrak{T}_H * \mathfrak{T}_H$ of $\mathfrak{T}_{H+1}$. This gives us an approximation to $P_{\vec{d}}(H+1)$ obtained by leaving out the variables:

$$P_{\vec{d}}(H * H) := (\beta_{\vec{d},0}(H * H), \beta_{\vec{d},1}(H * H), \dots, \beta_{\vec{d},n-1}(H * H)) \in \mathbb{P}_n.$$

Using the fact that $\mathfrak{T}_{H+1} = (\mathfrak{T}_H * \mathfrak{T}_H) \cup \mathfrak{T}_1$ and the weighting factors ι_H and o_H from Lemma 11, we have, for each $h < n$,

$$\begin{aligned} \beta_{\vec{d},h}(H+1) &= \operatorname{Prob}\langle t(\vec{d}) = h : t \in \mathfrak{T}_{H+1}\rangle \\ &= \frac{\|\mathfrak{T}_H * \mathfrak{T}_H\|}{\|\mathfrak{T}_{H+1}\|} \operatorname{Prob}\langle t(\vec{d}) = h : t \in \mathfrak{T}_H * \mathfrak{T}_H\rangle \\ &\quad + \frac{\|\mathfrak{T}_1\|}{\|\mathfrak{T}_{H+1}\|} \operatorname{Prob}\langle t(\vec{d}) = h : t \in \mathfrak{T}_1\rangle \\ &= \iota_H \beta_{\vec{d},h}(H * H) + o_H \beta_{\vec{d},h}(1) \end{aligned}$$

and therefore $P_{\vec{d}}(H+1) = \iota_H P_{\vec{d}}(H * H) + o_H P_{\vec{d}}(1)$. It turns out that $P_{\vec{d}}(H * H)$ is exactly the convolution.

Lemma 15. *For $H \in \mathbb{N}$ and $\vec{d} \in G^k$, we have $P_{\vec{d}}(H * H) = P_{\vec{d}}(H) \otimes P_{\vec{d}}(H)$.*

Proof. We compute the value of both distributions at $h \in G$.

$$\begin{aligned} P_{\vec{d}}(H * H)(h) &= \operatorname{Prob}\langle t(\vec{d}) = h : t \in \mathfrak{T}_H * \mathfrak{T}_H\rangle = \operatorname{Prob}\langle r(\vec{d})s(\vec{d}) = h : r, s \in \mathfrak{T}_H\rangle \\ &= \sum_{i*j=h} \operatorname{Prob}\langle r(\vec{d}) = i \quad \text{and} \quad s(\vec{d}) = j : r, s \in \mathfrak{T}_H\rangle \\ &= \sum_{i*j=h} \operatorname{Prob}\langle r(\vec{d}) = i : r \in \mathfrak{T}_H\rangle \cdot \operatorname{Prob}\langle s(\vec{d}) = j : s \in \mathfrak{T}_H\rangle \\ &= \sum_{i*j=h} \beta_{\vec{d},i}(H)\beta_{\vec{d},j}(H) = (P_{\vec{d}}(H) \otimes P_{\vec{d}}(H))(h). \end{aligned}$$

□

This gives us the recursion we need to test for asymptotic completeness.

Theorem 16. *Let $H \in \mathbb{N}$, let $k \in \mathbb{N}^{+1}$ and let $\vec{d} \in G^k$. Then*

(i) $P_{\vec{d}}(1) = \left(\frac{\|\vec{d}^{-1}(0)\|}{k}, \frac{\|\vec{d}^{-1}(1)\|}{k}, \dots, \frac{\|\vec{d}^{-1}(n-1)\|}{k}\right)$,
(ii) $P_{\vec{d}}(H+1) = \iota_H P_{\vec{d}}(H) \otimes P_{\vec{d}}(H) + o_H P_{\vec{d}}(1)$.

Proof. We obtain (i) from the uniform probability measure on $\mathfrak{T}_1$ and (ii) from Lemma 15 and the calculation preceding it. □

We used Theorem 16 to write a spreadsheet program that takes as input the table for $\mathbf{G}$, a value for k, and not $\vec{d}$ itself but rather the initial probability distribution $P_{\vec{d}}(1)$ defined by $\vec{d}$. It then iteratively generates the sequence $P_{\vec{d}}$ of probability distributions. We describe here the result of this computation with $\mathbf{G} = \mathbf{Pi}$, with $k = 10$, and with initial distribution $P_{\vec{d}} = (0.2, 0.1, 0.1, 0.5, 0.1)$. The asymptotic behavior we observed is typical of that of $\mathbf{A}_1$, $\mathbf{B}_1$, and most other randomly generated groupoids we have examined.

Example 17. Iterations to $H = 500$ indicate that $\mathbf{Pi}$ is asymptotically 10-complete.

Justification: In Fig. 5, we see that $\mathbf{Pi}$ has exactly two proper subgroupoids with underlying sets $\{1\}$ and $\{2\}$. If $e \in \{1, 2\}$, let $\vec{e} := (e, e, e, e, e, e, e, e, e, e) \in Pi^{10}$. Since e is idempotent, $t(\vec{e}) = e$ for every $t \in \mathcal{T}$. Consequently, for every $H \in \mathbb{N}$, we have $P_{\vec{1}}(H) = (0, 1, 0, 0, 0)$ and $P_{\vec{2}}(H) = (0, 0, 1, 0, 0)$. Thus $\underline{\mathbf{K}}_{\vec{e}} = \{e\} = \mathbf{K}_{\vec{e}}$ for $\vec{e} \in \{\vec{1}, \vec{2}\}$.

Now consider $\vec{d} \in Pi^{10} \backslash \{\vec{1}, \vec{2}\}$. Then $\mathbf{K}_{\vec{d}} = \mathbf{Pi}$. In Fig. 7, we give sample output for the initial distribution $P_{\vec{d}}(1) = (0.2, 0.1, 0.1, 0.5, 0.1)$ and $k = 10$. What we find is that these distributions become stable to six decimal places at $H = 16$ and remain so through $H = 500$. We summarize this assertion by writing

$$\mathbf{Pi} : (0.2, 0.1, 0.1, 0.5, 0.1)[16].$$

This one trial suggests that the $\beta_{\vec{d},a}$ sequences for $\mathbf{Pi}$ are not only all bounded away from 0, but that they actually converge. Three more trials with $k = 10$ not only give similar output, but all appear to converge to the same limit. Their summary data is given by

$$\mathbf{Pi} : (0.3, 0.2, 0.4, 0, 0.1)[17] \quad (0, 0.5, 0.5, 0, 0)[18] \quad (0, 0, 0, 0, 1)[21].$$

In contrast, the distribution sequence for input $\vec{1}$ produces the distribution sequence $P_{\vec{1}}(H) = (0, 1, 0, 0, 0)$ for every $H \in \mathbb{N}$. In order to do a more exacting

Pi	H	$\beta_{\vec{d},0}(H)$	$\beta_{\vec{d},1}(H)$	$\beta_{\vec{d},2}(H)$	$\beta_{\vec{d},3}(H)$	$\beta_{\vec{d},4}(H)$
	1	0.2	0.1	0.1	0.5	0.1
	2	0.127273	0.127273	0.145455	0.263636	0.336364
	4	0.099025	0.183535	0.088076	0.369279	0.260085
	6	0.102464	0.195934	0.089296	0.367367	0.244938
	8	0.103372	0.195554	0.091374	0.365539	0.244161
	10	0.103393	0.195290	0.091628	0.365402	0.244287
	16	0.103372	0.195275	0.091609	0.365434	0.244310
	500	0.103372	0.195275	0.091609	0.365434	0.244310

Fig. 7. Convergence of a distribution sequence for **Pi** with $k = 10$.

test, we tried the initial distribution $P_{\vec{d}}(1) = (0, 0.99999, 0, 0, 0.00001)$. In order to be able to realize this distribution with a $\vec{d} \in Pi^k$, we need to take $k = 10^5$. This $\vec{d}$ gives $\mathbf{K}_{\vec{d}} = \mathbf{Pi}$ and a substitution of 99.999% ones and 0.001% fours. Although this input distribution is very close to $(0, 1, 0, 0, 0)$, the output limit is identical to that of the other four trials with $\mathbf{K}_{\vec{d}} = \mathbf{Pi}$, stabilizing to six decimal places at $H = 33$:

$$\mathbf{Pi} : (0, 0.99999, 0, 0, 0.00001)[33].$$

Altogether we take this as strong evidence that $\mathbf{Pi}$ is asymptotically k-complete for all $k \in \mathbb{N}^{+1}$. $\square$

Murskiĭ's Theorem [5] tells us that the proportion of n-element groupoids with no non-trivial proper subgroupoids approaches 1 as n approaches infinity. In this sense $\mathbf{Pi}$ is a typical groupoid. Figure 8 shows a contrasting example of the atypical 5-element groupoid $\mathbf{S}$ that was deliberately constructed to have the five subgroupoids shown in the middle diagram. Each $k \in \mathbb{N}^{+1}$ and $\vec{d} \in S^k$ gives $\mathbf{K}_{\vec{d}}$ as one of these five subgroupoids. All spreadsheet runs for $\mathbf{S}$ appeared to converge on all inputs, looking similar to Fig. 7. The asymptotic residue of each subgroupoid in the center is shown on the left. Because we have $\{\underline{0, 1, 2}\} = \{1\}$ and $\{\underline{0, 1, 2, 3, 4}\} = \{1, 3, 4\}$, we see that $\mathbf{S}$ is not asymptotically k-complete for any $k \in \mathbb{N}^{+1}$.

4. Continuity

Throughout this section, we will consider a fixed but arbitrary choice of $k \in \mathbb{N}^{+1}$, a mutation set $\mathcal{V}$ and an n-element groupoid $\mathbf{G}$. Our goal is to show that $\mathbf{G}$ is k-term continuous if it is asymptotically complete and its subgroupoids have no separating relations. We will establish term continuity by proving condition (ii) of Theorem 3. Relative to this choice of k and $\mathbf{G}$, and a fixed choice of $\vec{d} \in G^k$, we abbreviate

$$\Pr \langle H \rangle := \operatorname{Prob} \langle t(\vec{d}, u(\vec{d})) \neq t(\vec{d}, v(\vec{d})) : (t, u, v) \in \mathcal{M}_H \rangle.$$

In this notation, we must show that $\lim_{H \to \infty} \Pr \langle H \rangle = 0$ for all $\vec{d} \in G^k$.

Let $\vec{z} = (z_1, z_2, z_3, \ldots)$ be a sequence of new variables. We define a *nested* $\Diamond$*-term* recursively. The variable $\Diamond$ is a nested $\Diamond$-term of height 1. If $w(\vec{z}, \Diamond)$ is a nested

$*$	0	1	2	3	4
0	1	2	2	3	1
1	0	1	1	4	3
2	2	1	0	1	3
3	3	4	0	3	4
4	4	4	0	1	3

$\mathbf{S} := \langle \{0, 1, 2, 3, 4\}, * \rangle$

$$\begin{tikzcd}[arrows=dash]
 & \{0,1,2,3,4\} \arrow[dl] \arrow[dr] & \\
\{0,1,2\} \arrow[d] & & \{1,3,4\} \arrow[dll] \arrow[d] \\
\{1\} & & \{3\}
\end{tikzcd}$$

Subgroupoids

$$\begin{tikzcd}[arrows=dash]
 & \{1,3,4\} \arrow[dl] \arrow[dr] & \\
\{1\} \arrow[d] & & \{1,3,4\} \arrow[dll] \arrow[d] \\
\{1\} & & \{3\}
\end{tikzcd}$$

Asymptotic residues

Fig. 8. An asymptotically incomplete groupoid.

$\Diamond$-term of height m, then $z_m w(\vec{z}, \Diamond)$ and $w(\vec{z}, \Diamond) z_m$ are nested $\Diamond$-terms of height $m+1$. We let $\mathcal{Z}_m$ denote the set of all 2^{m-1} nested $\Diamond$-terms of height m. Note that the height of a nested $\Diamond$-term is the depth of its $\Diamond$. For example,

$$(z_4((z_2(z_1\Diamond))z_3))z_5 \in \mathcal{Z}_6$$

is a nested $\Diamond$-term of height 6 with $\Diamond$ at depth 6. The lack of separating relations on a subgroupoid of $\mathbf{G}$ has an important implication for nested $\Diamond$-terms.

Non-Separation Lemma 18. *Assume that the subgroupoid* $\mathbf{K}$ *of* $\mathbf{G}$ *has no separating relation and let* $a, b \in K$ *with* $a \neq b$. *Then there exist* $m < n^2$, *a nested* $\Diamond$*-term* $s(\vec{z}, \Diamond)$ *of height* m *and* $\vec{c} \in K^{m-1}$ *such that* $s(\vec{c}, a) = s(\vec{c}, b)$.

Proof. Define

$$\sigma := \{(w(\vec{c}, a), w(\vec{c}, b)) \mid w(\vec{z}, \Diamond) \in \mathcal{Z}_m \text{ and } \vec{c} \in K^{m-1} \text{ for some } m \in \mathbb{N}\}.$$

Clearly σ satisfies parts (i) and (iii) of the definition of a separating relation on $\mathbf{K}$. Since $\mathbf{K}$ has no separating relation, it must not satisfy (ii). Thus there is an $m \in \mathbb{N}$, a $\vec{c} \in K^{m-1}$ and a nested $\Diamond$-term $s(\vec{z}, \Diamond)$ of height m such that $s(\vec{c}, a) = s(\vec{c}, b)$. Let m be the minimal height for which this is true. Consider all pairs $(w(\vec{c}, a), w(\vec{c}, b))$ where $w(\vec{z}, \Diamond)$ ranges over the distinct proper subterms of $s(\vec{z}, \Diamond)$, each of which is a nested $\Diamond$-term. Since m is minimal, these pairs must all be distinct. Consequently, there must be no more than $n^2 - n$ of them, so $m < n^2$. $\square$

For $H, m \in \mathbb{N}$, let $\mathcal{M}_{H,m}$ denote the set of all $(t, u, v) \in \mathcal{M}_H$ such that $\Diamond$ is at depth at least m in $t(\vec{x}, \Diamond)$. In particular, $\mathcal{M}_{H,m} \neq \varnothing$ if and only if $H \geq m$. For $H \geq m$, we give a construction that builds $\mathcal{M}_{H,m}$ from $\mathcal{M}_{H-m+1}$. Define the map

$$T : \mathcal{Z}_m \times \mathcal{T}_{H-m+1} \times \mathcal{T}_{H-m+2} \times \cdots \times \mathcal{T}_{H-1} \times \mathcal{M}_{H-m+1} \to \mathcal{M}_{H,m}$$

as follows. For $w \in \mathcal{Z}_m$, each $r_i \in \mathcal{T}_{H-m+i}$ and $(t', u, v) \in \mathcal{M}_{H-m+1}$, let

$$T(w, r_1, r_2, \ldots, r_{m-1}, (t', u, v)) := (t, u, v)$$

with

$$t(\vec{x}, \Diamond) := w(r_1(\vec{x}), r_2(\vec{x}), \ldots, r_{m-1}(\vec{x}), t'(\vec{x}, \Diamond)).$$

It will be helpful to also define the auxiliary $\Diamond$-term

$$w'(\vec{x}, \Diamond) := w(r_1(\vec{x}), r_2(\vec{x}), \ldots, r_{m-1}(\vec{x}), \Diamond),$$

giving us $t(\vec{x}, \Diamond) = w'(\vec{x}, t'(\vec{x}, \Diamond))$. (See Fig. 9.)

Lemma 19. *The map* T *is a bijection.*

Proof. Clearly a different choice of $(w, r_1, r_2, \ldots, r_{m-1}, (t', u, v))$ will yield a different $(t, u, v) \in \mathcal{M}_{H,m}$. Conversely, if we are given $(t, u, v) \in \mathcal{M}_{H,m}$, we can follow a path from the root of the term tree for $t(\vec{x}, \Diamond)$ toward $\Diamond$. At depth m we find the subterm $t'(\vec{x}, \Diamond)$ and we find the terms $r_{m-1}(\vec{x}), \ldots, r_2(\vec{x}), r_1(\vec{x})$ along the way. The path itself is the trunk of $w(\vec{z}, \Diamond)$. $\square$

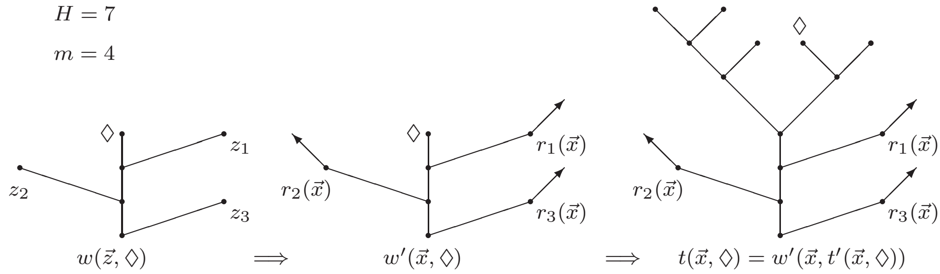


Fig. 9. $(t, u, v) \in \mathcal{M}_{7,4}$ from $w \in \mathcal{Z}_4$, $r_i \in \mathcal{T}_{3+i}$ and $(t', u, v) \in \mathcal{M}_4$.

We will now imagine a mutation (t, u, v) with $\Diamond$ at great depth in $t(\vec{x}, \Diamond)$, and then imagine substituting a sequence $\vec{d}$ for $\vec{x}$ in both $t(\vec{x}, u(\vec{x}))$ and $t(\vec{x}, v(\vec{x}))$. As we travel from the vertices of $u(\vec{x})$ and $v(\vec{x})$ in the term trees down toward the root, we see $u(\vec{d})$ and $v(\vec{d})$ multiplied from the left and right by the same sequence of elements of $\mathbf{G}$. We will investigate the probability that the two resulting values remain distinct as we move down the tree. Our goal is to show that, once the tree above us grows beyond some height H_0, this probability will decay exponentially to zero.

The key ideas all come together in our next lemma, where we will see directly how asymptotic completeness and lack of separating relations conspire to show why we might expect $\Pr\langle H\rangle$ to converge to zero. In order to prove this lemma we need to apply Lemma 19 with $m := n^2$, where n is the cardinality of $\mathbf{G}$. In what follows, we will denote the set $\mathcal{M}_{H,n^2}$ of mutations by $\mathcal{M}_{H,\mathbf{G}}$, bearing in mind that the only relevance of the groupoid $\mathbf{G}$ is its cardinality. If $w(\vec{x}, \Diamond)$ is a $\Diamond$-term, we will use the abbreviations

$$w[u] := w(\vec{d}, u(\vec{d})) \quad \text{and} \quad w[v] := w(\vec{d}, v(\vec{d})).$$

Since equality of $t'[u]$ and $t'[v]$ implies equality of $t[u]$ and $t[v]$, we will further restrict our attention to those mutations in $\mathcal{M}_{H,\mathbf{G}}$ where $t'[u] \neq t'[v]$. We then examine the probability that their output values will have collapsed by the time we reach the root of the tree.

Central Lemma 20. *Let* $\mathbf{G}$ *be an asymptotically* k*-complete finite groupoid whose subgroupoids have no separating relations. Then there is a* $\gamma > 0$ *and an* $H_1 \in \mathbb{N}$ *such that, for all* $H > H_1$ *and all* $\vec{d} \in G^k$,

$$\operatorname{Prob}\langle t[u] = t[v] : (t, u, v) \in \mathcal{M}_{H,\mathbf{G}} \quad \textit{and} \quad t'[u] \neq t'[v]\rangle > \gamma.$$

Proof. Since $\mathbf{G}$ is finite, there is a single choice of $\delta > 0$ and an $H_0 \in \mathbb{N}$ such that $\beta_{\vec{d},a}(H) > \delta$ for all $H > H_0$, all $\vec{d} \in G^k$ and all $a \in \underline{K}_{\vec{d}}$. Let $\gamma := (\delta/2)^{n^2} > 0$ and let $H_1 = H_0 + n^2$. Consider any $H > H_1$ and $\vec{d} \in G^k$, and let $(t, u, v) \in \mathcal{M}_{H,\mathbf{G}}$. We apply Lemma 19 to represent (t, u, v) as

$$T(w, r_1, r_2, \ldots, r_{n^2-1}, (t', u, v)) = (t, u, v).$$

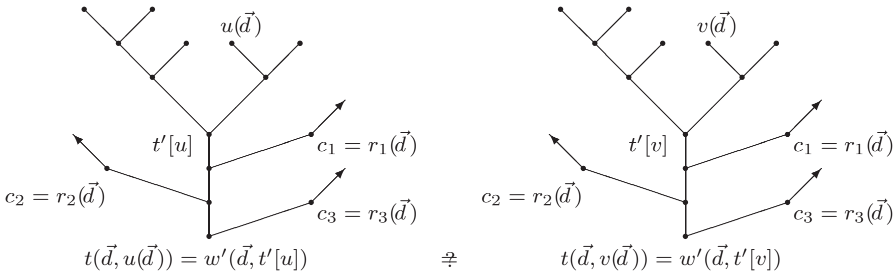


Fig. 10. Mutation replaces $u(\vec{d})$ with $v(\vec{d})$ in $t(\vec{d}, u(\vec{d}))$.

Figure 10 illustrates the substitution of $\vec{d}$ for $\vec{x}$ in $t(\vec{x}, u(\vec{x}))$ and $t(\vec{x}, v(\vec{x}))$ and the subsequent path down the term tree.

Assume that $t'[u] \neq t'[v]$ and take $\mathbf{K} := \mathbf{K}_{\vec{d}}$ in Lemma 18. Since $t'[u]$ and $t'[v]$ are in K and $\mathbf{K}$ has no separating relation, we see from Lemma 18 that there is a $\vec{c} \in K^{n^2-1}$ and a nested $\Diamond$-term $s(\vec{z}, \Diamond) \in \mathcal{Z}_{n^2}$ such that $s(\vec{c}, t'[u]) = s(\vec{c}, t'[v])$. Since $\mathbf{G}$ is asymptotically k-complete, we have $\underline{K}_{\vec{d}} = K$ so that each c_i is in $\underline{K}_{\vec{d}}$. Since each $r_i(\vec{x})$ is chosen from $\mathcal{T}_{H-n^2+i}$ with

$$H - n^2 + i > H - n^2 > H_1 - n^2 = H_0,$$

the probability $\beta_{\vec{d},c_i}(H - n^2 + i)$ that $r_i(\vec{d}) = c_i$ is greater than δ. And since $\|\mathcal{Z}_{n^2}\| = 2^{n^2-1}$, the probability that $w(\vec{z}, \Diamond) = s(\vec{z}, \Diamond)$ is greater than 2^{-n^2}. Since $r_1, \ldots, r_{n^2-1}$ and w are chosen independently,

$$\operatorname{Prob}\langle w = s \text{ and } r_i(\vec{d}) = c_i \text{ for } i = 1, 2, \ldots, n^2 - 1\rangle > 2^{-n^2}\delta^{n^2} = \left(\frac{\delta}{2}\right)^{n^2} = \gamma.$$

If it is true that $w = s$ and $r_i(\vec{d}) = c_i$ for $i = 1, 2, \ldots, n^2 - 1$, then

$$\begin{aligned} t(\vec{d}, u(\vec{d})) &= w'(\vec{d}, t'[u]) = w(r_1(\vec{d}), \ldots, r_{n^2-1}(\vec{d}), t'[u]) \\ &= w(\vec{c}, t'[u]) = s(\vec{c}, t'[u]) = s(\vec{c}, t'[v]) = w(\vec{c}, t'[v]) \\ &= w(r_1(\vec{d}), \ldots, r_{n^2-1}(\vec{d}), t'[v]) = w'(\vec{d}, t'[v]) = t(\vec{d}, v(\vec{d})). \end{aligned}$$

Consequently, we have

$$\operatorname{Prob}\langle t(\vec{d}, u(\vec{d})) = t(\vec{d}, v(\vec{d})) : (t, u, v) \in \mathcal{M}_{H,\mathbf{G}} \text{ and } t'[u] \neq t'[v]\rangle > \gamma$$

as required. $\square$

This lemma appears to say something about the convergence of

$$\Pr\langle H\rangle = \operatorname{Prob}\langle t[u] \neq t[v] : (t, u, v) \in \mathcal{M}_H\rangle$$

to zero, but it is limited by considering only mutations in the subset $\mathcal{M}_{H,\mathbf{G}}$ of $\mathcal{M}_H$. The relevance of the Central Lemma will therefore depend on knowing if $\mathcal{M}_{H,\mathbf{G}}$ constitutes a sufficiently large part of $\mathcal{M}_H$. The next two lemmas concern only groupoid terms, and make no mention of a specific groupoid other than its cardinality.

For $H \in \mathbb{N}$, recall from Sec. 1 that $\mathcal{M}_H = \mathcal{U}_H \times \mathcal{V}$. We denote by

$$\mu_H := \frac{\|\mathcal{U}_H\|}{\|\mathcal{T}_H\|}$$

the average number of subterms of t as t ranges over $\mathcal{T}_H$. A moment's thought suggests that an average term of height $H+1$ should have about twice as many subterms as a term of height H, plus itself. This idea is made exact in part (i) of the next lemma. Recall from Sec. 3 that $\iota_H := \|\mathcal{T}_H\|^2/(\|\mathcal{T}_H\|^2 + k)$ and that this sequence converges very rapidly to 1. If $m \in \mathbb{N}$, let $\mathcal{U}_{H,m}$ be the set of all $(t(\vec{x}, \Diamond), u(\vec{x})) \in \mathcal{U}_H$ with $\Diamond$ at depth at least m in $t(\vec{x}, \Diamond)$.

Lemma 21. *For all $H, m \in \mathbb{N}$ and $k \in \mathbb{N}^{+1}$,*

(i) $$\mu_{H+1} = 2\mu_H \iota_H + 1,$$

(ii) $$\lim_{H\to\infty} \mu_H = \infty,$$

(iii) $$\lim_{H\to\infty} \frac{\|\mathcal{U}_{H,m}\|}{\|\mathcal{U}_H\|} = 1.$$

Proof. We prove (i), and then show (i) implies (ii) and (ii) implies (iii). First, let

$$u_H := \|\mathcal{U}_H\|, \quad u_{H,m} := \|\mathcal{U}_{H,m}\|, \quad t_H := \|\mathcal{T}_H\|.$$

We will use five easily established facts:

(1) $t_1 = k$ and $t_{H+1} = t_H^2 + k$.
(2) $u_1 = k$ and $u_H = t_H \mu_H$.
(3) $u_{H,1} = u_H$.
(4) A bijection from $\{L, R\} \times \mathcal{T}_H \times \mathcal{U}_{H,m}$ to $\mathcal{U}_{H+1,m+1}$ is given by

$$(L, r, (t, u)) \mapsto (t^\sharp, u) \quad \text{with } t^\sharp(\vec{x}, \Diamond) := r(\vec{x})t(\vec{x}, \Diamond),$$

$$(R, r, (t, u)) \mapsto (t^\flat, u) \quad \text{with } t^\flat(\vec{x}, \Diamond) := t(\vec{x}, \Diamond)r(\vec{x}).$$

(5) $u_{H+1,m+1} = 2t_H u_{H,m}$.

(i) If $H \in \mathbb{N}$, then $\mathcal{U}_{H+1} = \mathcal{U}_{H+1,2} \cup \{(\Diamond, t) \mid t \in \mathcal{T}_{H+1}\}$. Counting pairs, we have

$$u_{H+1} = u_{H+1,2} + t_{H+1} = 2t_H u_{H,1} + t_{H+1} = 2t_H u_H + t_{H+1},$$

$$\mu_{H+1} = \frac{u_{H+1}}{t_{H+1}} = \frac{2t_H u_H + t_{H+1}}{t_{H+1}} = 2\frac{t_H}{t_H^2 + k}(t_H \mu_H) + 1 = 2\mu_H \iota_H + 1.$$

(ii) For all $H \in \mathbb{N}$, we have

$$\iota_H = \frac{t_H^2}{t_H^2 + k} \geq \frac{k^2}{k^2 + k} = \frac{k}{k+1} \geq \frac{2}{2+1} = \frac{2}{3}.$$

Applying (i), we conclude that

$$\mu_{H+1} = 2\mu_H \iota_H + 1 > \frac{4}{3}\mu_H \quad \text{and therefore } \mu_H \geq (4/3)^{H-1} \to \infty.$$

(iii) We use induction on m. If $m = 1$ it is true because $\mathcal{U}_{H,1} = \mathcal{U}_H$, and therefore $\|\mathcal{U}_{H,1}\|/\|\mathcal{U}_H\| = u_H/u_H = 1$. Assuming it is true for m, we use the above facts to calculate

$$\begin{aligned}\frac{\|\mathcal{U}_{H+1,m+1}\|}{\|\mathcal{U}_{H+1}\|} &= \frac{u_{H+1,m+1}}{u_{H+1}} = \frac{2t_H u_{H,m}}{t_{H+1}\mu_{H+1}} = \frac{u_{H,m}}{u_H} \cdot \frac{2t_H u_H}{t_{H+1}\mu_{H+1}} \\ &= \frac{u_{H,m}}{u_H} \cdot \frac{2t_H t_H \mu_H}{(t_H^2 + k)\mu_{H+1}} = \frac{u_{H,m}}{u_H} \cdot \frac{2\iota_H \mu_H}{\mu_{H+1}} \\ &= \frac{\|\mathcal{U}_{H,m}\|}{\|\mathcal{U}_H\|} \cdot \frac{\mu_{H+1} - 1}{\mu_{H+1}} \to 1 \cdot 1 = 1,\end{aligned}$$

since the first factor goes to one by induction and the second goes to one by (ii). □

Corollary 22. *For all $m \in \mathbb{N}$,*

$$\lim_{H \to \infty} \frac{\|\mathcal{M}_{H,m}\|}{\|\mathcal{M}_H\|} = 1.$$

Proof. For all $m \in \mathbb{N}$, Lemma 21 gives us

$$\frac{\|\mathcal{M}_{H,m}\|}{\|\mathcal{M}_H\|} = \frac{\|\mathcal{U}_{H,m} \times \mathcal{V}\|}{\|\mathcal{U}_H \times \mathcal{V}\|} = \frac{\|\mathcal{U}_{H,m}\|}{\|\mathcal{U}_H\|} \to 1.$$

□

Given that $\mathcal{M}_{H,\mathbf{G}}$ asymptotically constitutes almost all of $\mathcal{M}_H$, we will now use the Central Lemma 20 to show that $\Pr\langle H \rangle$ converges to zero.

Lemma 23. *Let $\mathbf{G}$ be an asymptotically k-complete finite groupoid whose subgroupoids have no separating relations. Then there is a positive $\rho < 1$ and an $H_2 \in \mathbb{N}$ such that, for all $\vec{d} \in G^k$ and all $J > H_2$,*

$$\Pr\langle J + n^2 - 1\rangle < \rho \Pr\langle J\rangle.$$

Proof. Choose H_1 and γ as in the Central Lemma 20 and let $\rho := 1/(1+\gamma)$. Choose $H_2 \geq H_1$ so that, if $J > H_2$, then $\|M_{J,\mathbf{G}}\|/\|\mathcal{M}_J\| > \rho$. Now let $J > H_2$. Applying Lemma 19 with $m := n^2$ and $H := J + n^2 - 1$, we have the bijection

$$T : \mathcal{Z}_{n^2} \times \cdots \times \mathcal{M}_J \to \mathcal{M}_{J+n^2-1,n^2} = \mathcal{M}_{J+n^2-1,\mathbf{G}}.$$

Define

$$S : \mathcal{M}_{J+n^2-1,\mathbf{G}} \to \mathcal{M}_J$$

as T^{-1} followed by the projection onto the last factor. Then S maps (t, u, v) to (t', u, v) where $t(\vec{x}, \Diamond) = w'(\vec{x}, t'(\vec{x}, \Diamond))$ for some $\Diamond$-term w'. From Lemma 19, we know that all $(t', u, v) \in \mathcal{M}_J$ have the same number of pre-images in $\mathcal{M}_{J+n^2-1,\mathbf{G}}$

under S. It follows that, if $S(t,u,v) = (t',u,v)$, then

$$\begin{aligned}&\operatorname{Prob}\langle t'[u] = t'[v] : (t,u,v) \in \mathcal{M}_{J+n^2-1,\mathbf{G}}\rangle\\ &\quad = \operatorname{Prob}\langle t'[u] = t'[v] : (t',u,v) \in \mathcal{M}_J\rangle = 1 - \operatorname{Pr}\langle J\rangle.\end{aligned}$$

Using this fact and the fact that $J + n^2 - 1 > J > H_2 \geq H_1$, we have

$$\begin{aligned}1 - \operatorname{Pr}\langle J + n^2 - 1\rangle &\geq \frac{\|\mathcal{M}_{J+n^2-1,\mathbf{G}}\|}{\|\mathcal{M}_{J+n^2-1}\|} \cdot \operatorname{Prob}\langle t[u] = t[v] : (t,u,v) \in \mathcal{M}_{J+n^2-1,\mathbf{G}}\rangle\\ &> \rho(\operatorname{Prob}\langle t'[u] = t'[v] : (t,u,v) \in \mathcal{M}_{J+n^2-1,\mathbf{G}}\rangle\\ &\quad + \operatorname{Pr}\langle t[u] = t[v] : (t,u,v) \in \mathcal{M}_{J+n^2-1,\mathbf{G}}, t'[u] \neq t'[v]\rangle)\\ &> \rho(1 - \operatorname{Pr}\langle J\rangle + \gamma)\\ &= \rho(1+\gamma) - \rho\operatorname{Pr}\langle J\rangle\\ &= 1 - \rho\operatorname{Pr}\langle J\rangle.\end{aligned}$$

Consequently $\operatorname{Pr}\langle J + n^2 - 1\rangle < \rho\operatorname{Pr}\langle J\rangle$. □

Continuity Theorem 24. *Let $k \in \mathbb{N}^{+1}$ and let $\mathbf{G}$ be a finite groupoid that is asymptotically k-complete. Then $\mathbf{G}$ is k-term continuous if and only if no subgroupoid of $\mathbf{G}$ has a separating relation.*

Proof. If $\mathbf{G}$ is k-term continuous, then no subgroupoid of $\mathbf{G}$ has a separating relation by Theorem 6. Conversely, assume that no subgroupoid of $\mathbf{G}$ has a separating relation. Let $\vec{d} \in G^k$. Partition the sequence $\{\operatorname{Pr}\langle H\rangle \mid H \in \mathbb{N}\}$ into $n^2 - 1$ subsequences $S_0, S_1, \ldots, S_{n^2-2}$ with $S_i(m) := \operatorname{Pr}\langle i + m(n^2-1)\rangle$ for $m \in \mathbb{N}$. Then Lemma 23 tells us that each subsequence converges to zero and consequently $\operatorname{Pr}\langle H\rangle$ converges to zero as well. By Theorem 3(ii), the groupoid $\mathbf{G}$ is k-term continuous. □

5. Examples

If our spreadsheet data accurately represents the asymptotic behavior of the groupoid $\mathbf{Pi}$, then Example 17 shows that $\mathbf{Pi}$ is k-term continuous for each $k \in \mathbb{N}^{+1}$. This example is typical of many other randomly generated groupoids we have tested, suggesting that most finite groupoids are term continuous. Rather than presenting more similar examples, in this section we will present several different examples that answer some natural questions and open others. Altogether there appears to be much remaining to ask and learn about this subject.

We start with two examples that are simple enough for us to *prove* that they are term continuous without relying on experimental data. The first is an application of the Continuity Theorem 24.

Example 25. Let $\mathbf{L} := \langle\{0,1,\ldots,n-1\},*\rangle$ be a left-zero semigroup with $a*b := a$ for all $a,b \in L$. Then $\mathbf{L}$ is k-term continuous for every $k \in \mathbb{N}^{+1}$.

Proof. Let $\sigma \subseteq L \times L$ satisfy conditions (i) and (iii) of a separating relation, and choose $(a, b) \in \sigma$. Then $(a, a) = (a * a, a * b) \in \sigma$ so σ does not satisfy condition (ii) and is therefore not a separating relation.

Now let $k \in \mathbb{N}^{+1}$. To see that $\mathbf{L}$ is asymptotically k-complete, choose $\vec{d} \in L^k$. We will show that $\beta_{\vec{d},a}$ converges to $\|\vec{d}^{-1}(a)\|/k$ for every $a \in L$. Since $K_{\vec{d}} = \{d_0, d_1, \ldots, d_{k-1}\}$, it will follow that $\underline{K}_{\vec{d}} = K_{\vec{d}}$. Thus $\mathbf{L}$ will be asymptotically k-complete and therefore k-term continuous by the Continuity Theorem 24.

Let $H \in \mathbb{N}$. For $a \in L$, let $\mathfrak{X}_{\vec{d},a}$ be the set of terms in $\mathfrak{T}_H$ which begin with a variable x_i for which $d_i = a$. Then $\mathfrak{T}_H = \mathfrak{X}_{\vec{d},0} \cup \mathfrak{X}_{\vec{d},1} \cup \cdots \cup \mathfrak{X}_{\vec{d},n-1}$ with $\|\mathfrak{X}_{\vec{d},a}\| = (\|\vec{d}^{-1}(a)\|/k)\|\mathfrak{T}_H\|$. For $a \in L$ and $t(\vec{x}) \in \mathfrak{T}_H$, we have $t(\vec{d}) = a$ if and only if $t(\vec{x}) \in \mathfrak{X}_{\vec{d},a}$. It follows that, for $a \in L$,

$$\beta_{\vec{d},a}(H) = \frac{\|\mathfrak{X}_{\vec{d},a}\|}{\|\mathfrak{T}_H\|} = \frac{\|\vec{d}^{-1}(a)\|}{k}.$$

Since this is true for every $H \in \mathbb{N}$, the sequence $\beta_{\vec{d},a}$ has constant value $\|\vec{d}^{-1}(a)\|/k$ and therefore converges to $\|\vec{d}^{-1}(a)\|/k$. Hence $\mathbf{L}$ is asymptotically k-complete and therefore k-term continuous. $\square$

Theorem 6 says that the absence of separating relations is a necessary condition for term continuity. Is asymptotic completeness also necessary? While the Continuity Theorem 24 uses asymptotic completeness to establish condition (ii) of Theorem 3, we will use condition (iii) of Theorem 3 to give an example that is term continuous but not asymptotically complete. Even this seemingly trivial example requires Theorem 3, Lemma 21 and the argument here to establish term continuity. This suggests that *proving* term continuity for more substantial examples may be difficult.

Example 26. Let $n, k \in \mathbb{N}^{+1}$ and let $\mathbf{C} := \langle\{0, 1, \ldots, n-1\}, *\rangle$ where $*$ has constant value 0. Then $\mathbf{G}$ is k-term continuous but is not asymptotically complete.

Proof. To see that $\mathbf{C}$ is not asymptotically complete, let $\vec{d} \in C^k$ and let $t(\vec{x}) \in \mathfrak{T}$. If $t(\vec{x}) \notin \mathfrak{T}_1$, then $t(\vec{d}) = 0$. Consequently,

$$\beta_{\vec{d},0}(H) = \mathrm{Prob}\,\langle t(\vec{d}) = 0 : t \in \mathfrak{T}_H\rangle \geq \frac{\|\mathfrak{T}_H \backslash \mathfrak{T}_1\|}{\|\mathfrak{T}_H\|} = \frac{\|\mathfrak{T}_H\| - k}{\|\mathfrak{T}_H\|} \to 1 \quad \text{as } H \to \infty.$$

Thus $\underline{K}_{\vec{d}} = \{0\}$. But if $\vec{d} \neq (0, 0, \ldots, 0)$, then $K_{\vec{d}} \neq \{0\}$.

We now show that $\mathbf{C}$ is term continuous. If $t(\vec{x}, u(\vec{x}))^{\mathbf{C}} \neq t(\vec{x}, v(\vec{x}))^{\mathbf{C}}$, then it must be that $t(\vec{x}, \Diamond) = \Diamond$. In this case,

$$\begin{aligned}&\mathrm{Prob}\,\langle t(\vec{x}, u(\vec{x}))^{\mathbf{C}} \neq t(\vec{x}, v(\vec{x}))^{\mathbf{C}} : (t, u, v) \in \mathfrak{M}_H\rangle \\ &\qquad\qquad \leq \mathrm{Prob}\,\langle t(\vec{x}, \Diamond) = \Diamond : (t, u, v) \in \mathfrak{M}_H\rangle\end{aligned}$$

$$= \frac{\|\{(\Diamond, u, v) \mid u \in \mathcal{T}_H, v \in \mathcal{V}\}\|}{\|\mathcal{M}_H\|} = \frac{\|\mathcal{T}_H\|\|\mathcal{V}\|}{\|\mathcal{U}_H\|\|\mathcal{V}\|}$$

$$= \frac{1}{\mu_H} \to 0 \quad \text{as } H \to \infty$$

by Lemma 21(ii). Hence

$$\lim_{H\to\infty} \operatorname{Prob}\langle t(\vec{x}, u(\vec{x}))^{\mathbf{C}} = t(\vec{x}, v(\vec{x}))^{\mathbf{C}} : (t, u, v) \in \mathcal{M}_H\rangle = 1,$$

and **C** is term continuous by Theorem 3(iii). □

What about primal groupoids? The original goal was to find terms for primal groupoids by evolution. Must a primal groupoid be term continuous?

Example 27. The 5-element quasigroup **Q** in Fig. 11 is primal but not k-term continuous for any $k \in \mathbb{N}^{+1}$. However, iterations to $H = 500$ indicate that **Q** is asymptotically 10-complete.

Justification: The quasigroup **Q** has no proper subalgebras, congruences or automorphisms and is therefore primal by Rouseau [8]. By Corollary 7, it is not k-term continuous for any $k \in \mathbb{N}^{+1}$. Its collapsing table in Fig. 11 confirms this fact with $\rho_\omega = \rho_1 \neq Q^2$, and shows that the separating relation $Q^2 \backslash \rho_\omega$ is $\neq$.

As we might expect from the symmetry of a quasigroup, computations of its probability distributions indicate that it is asymptotically complete by quickly converging to the uniform distribution. Figure 12 shows the beginning of a typical distribution sequence with six decimal place stability from $H = 6$ to $H = 500$. □

An interesting contrast with **Q** is the groupoid **NQ** in Fig. 13, obtained from **Q** by changing the value of $2 * 3$ from 1 to 2. While these two groupoids are very similar, they straddle the fence between having and not having a separating relation.

Example 28. Let **NQ** be the 5-element groupoid in Fig. 13. Then **NQ** is primal and has no separating relation. Iterations to $H = 500$ indicates that it is asymptotically 10-complete and therefore 10-term continuous.

$*$	0	1	2	3	4
0	1	4	0	2	3
1	3	2	4	0	1
2	2	0	3	1	4
3	0	3	1	4	2
4	4	1	2	3	0

$\mathbf{Q} := \langle\{0, 1, 2, 3, 4\}, *\rangle$

	0	1	2	3	4
0	ρ_1				
1		ρ_1			
2			ρ_1		
3				ρ_1	
4					ρ_1

Fig. 11. Collapsing table for the primal quasigroup **Q**.

Q	H	$\beta_{\vec{d},0}(H)$	$\beta_{\vec{d},1}(H)$	$\beta_{\vec{d},2}(H)$	$\beta_{\vec{d},3}(H)$	$\beta_{\vec{d},4}(H)$
	1	0.2	0.4	0.0	0.3	0.1
	2	0.136364	0.145455	0.227273	0.309091	0.181818
	3	0.183485	0.212139	0.191660	0.198266	0.214451
	4	0.200064	0.200653	0.200168	0.199774	0.199342
	5	0.200001	0.199999	0.200000	0.200000	0.200000
	6	0.200000	0.200000	0.200000	0.200000	0.200000
	500	0.200000	0.200000	0.200000	0.200000	0.200000

Fig. 12. Convergence of distribution for **Q** with $k = 10$.

$*$	0	1	2	3	4
0	1	4	0	2	3
1	3	2	4	0	1
2	2	0	3	2	4
3	0	3	1	4	2
4	4	1	2	3	0

NQ $:= \langle\{0, 1, 2, 3, 4\}, *\rangle$

	0	1	2	3	4
0	ρ_1	$\rho_3\|2*$	$\rho_2\|*3$	$\rho_2\|2*$	$\rho_3\|3*$
1		ρ_1	$\rho_3\|*1$	$\rho_3\|1*$	$\rho_3\|*3$
2			ρ_1	$\rho_3\|0*$	$\rho_3\|0*$
3				ρ_1	$\rho_3\|*4$
4					ρ_1

Fig. 13. Collapsing table for **NQ** : $\rho_\omega = \rho_3 = NQ^2$.

NQ	H	$\beta_{\vec{d},0}(H)$	$\beta_{\vec{d},1}(H)$	$\beta_{\vec{d},2}(H)$	$\beta_{\vec{d},3}(H)$	$\beta_{\vec{d},4}(H)$
	1	0.2	0.4	0.0	0.3	0.1
	4	0.195439	0.149213	0.252862	0.203858	0.198628
	7	0.194399	0.148708	0.254284	0.205199	0.197410
	10	0.194371	0.148672	0.254338	0.205219	0.197401
	13	0.194369	0.148670	0.254340	0.205220	0.197400
	500	0.194369	0.148670	0.254340	0.205220	0.197400

Fig. 14. Convergence of distribution for **NQ** with $k = 10$.

Justification: Again using Rouseau [8] we see that **NQ** is primal. Its collapsing table in Fig. 13 shows that it does not have a separating relation, as $\rho_\omega = \rho_3 = NQ^2$.

Figure 14 indicates that the replacement of only a single 1 with a 2 in the table for **Q** makes a correspondingly small change in the asymptotic behavior: a slightly lower probability for 1 and higher for 2, with little change otherwise. These iterations begin with the same input distribution as used for **Q** and have six decimal place stability from $H = 13$ to $H = 500$. □

Is every primal groupoid asymptotically complete? Many seem to be. As a final example, consider again the smallest primal groupoid, Sheffer's NAND groupoid **S**, shown in Fig. 1. We can test this groupoid for asymptotic completeness with a simple hand calculation. Start with a large term of 2^m variable occurrences, all at depth $m + 1$, and randomly assign 0s and 1s to the variables. Now compute from

H	$\beta_{\vec{d},0}(H)$	$\beta_{\vec{d},1}(H)$	H	$\beta_{\vec{d},0}(H)$	$\beta_{\vec{d},1}(H)$	H	$\beta_{\vec{d},0}(H)$	$\beta_{\vec{d},1}(H)$
1	0.500000	0.500000	8	0.105541	0.894459	15	0.999700	0.000300
2	0.285714	0.714286	9	0.800057	0.199943	16	0.000000	1.000000
3	0.510169	0.489831	10	0.039977	0.960023	17	1.000000	0.000000
4	0.239934	0.760066	11	0.921644	0.078356	18	0.000000	1.000000
5	0.577700	0.422300	12	0.006140	0.993860	19	1.000000	0.000000
6	0.178338	0.821662	13	0.987758	0.012242	20	0.000000	1.000000
7	0.675129	0.324871	14	0.000150	0.999850	21	1.000000	0.000000

Fig. 15. Oscillating distribution for Sheffer's **S** with $k = 6$.

top down. A minority of adjacent pairs will be assigned $(1, 1)$, so most values at depth m will be 1. With most adjacent pairs at depth m being 1, most values at depth $m - 1$ will be 0. Continuing this way suggests that the sequences $\beta_{\vec{d},0}$ and $\beta_{\vec{d},1}$ will each alternately approach 0 and 1. (Try it!)

This expectation is borne out by the simulation in Fig. 15, which appears to be a dramatic failure of asymptotic completeness through the oscillatory behavior required by Corollary 14. Iterations to $H = 500$ show no change in this pattern within six decimal places. Other examples suggest that this kind of oscillatory behavior is uncommon but not rare among very small groupoids.

6. Conclusions

We have given testable conditions under which a finite groupoid is term continuous, opening a multitude of avenues for further study. We state here a mix of interesting questions, problems and conjectures. In the upcoming study of Clark, Keijzer and Spector [2], we will present two evolutionary algorithms and show them capable of finding terms representing arbitrarily given operations in truly huge search spaces. We will provide compelling evidence that these algorithms require term continuity, and that they are successful for almost all finite term continuous groupoids.

Everything in this paper is done in the context of finite algebras with a single binary operation, including some detailed analyses of their associated terms. It is therefore not obvious in advance that these results extend to other similarity types.

Problem 29. *Find types other than $\langle 2 \rangle$ for which term continuity can be similarly defined and established.*

Almost all of the asymptotically complete groupoids we have seen appear to have a much stronger property that is illustrated by the groupoid **Pi**, yet we have no explanation as to why this should be true.

Problem 30. *Find conditions on a finite groupoid* **G** *that imply, for all* $k \in \mathbb{N}^{+1}$,

(i) $P_{\vec{d}}$ *converges for every* $\vec{d} \in G^k$, *and*
(ii) *if* $\vec{d}, \vec{d}' \in G^k$ *and* $K_{\vec{d}} = K_{\vec{d}'}$, *then* $\lim_{H\to\infty} P_{\vec{d}}(H) = \lim_{H\to\infty} P_{\vec{d}'}(H)$.

When **G** has these properties, these limiting distributions appear to be interesting invariants of its subgroupoids.

We have only been able to prove term continuity in the simplest possible cases: a constant groupoid and a left-(or right-)zero groupoid.

Problem 31. *Find other finite groupoids that are provably term continuous.*

Our Separating Relations Test 9 gave an efficient decision method for the presence of separating relations.

Question 32. Is asymptotic completeness a decidable property of finite groupoids?

Question 33. Is term continuity a decidable property of finite groupoids?

In contrast to the seeming difficulty of proving a particular groupoid term continuous, our data gives experimental evidence that randomly constructed finite groupoids normally are. We say that almost all finite groupoids have a property P if the proportion of groupoids over $\{0, 1, 2, \ldots, n-1\}$ having property P approaches 1 as n approaches infinity. For example, Murskiĭ's theorem [5] implies that almost all finite groupoids are idemprimal.

Question 34. For $k \in \mathbb{N}^{+1}$, are almost all finite groupoids k-term continuous?

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